The Gauss map of submanifolds in the Heisenberg group

E.V. Petrov

Karazin Kharkiv National University, 4 Svobody sq., 61077 Kharkiv, Ukraine

---

Abstract

We obtain criteria for the harmonicity of the Gauss map of submanifolds in the Heisenberg group and consider the examples demonstrating the connection between the harmonicity of this map and the properties of the mean curvature field. Also, we introduce a natural class of cylindrical submanifolds and prove that a constant mean curvature hypersurface with harmonic Gauss map is cylindrical.

Keywords: Heisenberg group, Gauss map, harmonic map, mean curvature field, constant mean curvature hypersurface
2000 MSC: 53C40, 53C42, 53C43

---

1. Introduction

It was proved in [1] that the Gauss map of a submanifold in the Euclidean space is harmonic (as a map of Riemannian manifolds, see [2]) if and only if the mean curvature field of this submanifold is parallel. There exist a natural generalization of the Gauss map to submanifolds in a Lie group: for each point of a submanifold the tangent space at this point is translated to the identity element of the group. Let the Lie group be endowed with some left invariant metric. In [3] it was proved that when this metric is biinvariant and the submanifold is a hypersurface the Gauss map is harmonic if and only if the mean curvature is constant.

In [4] we obtained a harmonicity criterion for the Gauss map of a submanifold in a Lie group in the terms of the second fundamental form and the left invariant Levi-Civita connection (see Theorem 1 here). Earlier in [5] we obtained such criteria for hypersurfaces in 2-step nilpotent Lie groups and found that the constance of the mean curvature is not, in general, equivalent to the harmonicity of the Gauss map. In this paper we consider the Heisenberg groups that are the simplest nonabelian 2-step nilpotent Lie groups.

The paper is organized as follows. In Section 2 we give some preliminaries concerned the Gauss map, harmonic maps and the Heisenberg groups. In Section 3 we prove criteria for the harmonicity of the Gauss map of submanifolds in the Heisenberg group (Proposition 2) and give examples of minimal submanifolds with the nonharmonic Gauss map (Proposition 3). Section 4 is devoted to the study of cylindrical submanifolds in the Heisenberg groups. We call a submanifold cylindrical if it is tangent to the one-dimensional foliation generated by the center of the Lie algebra of the group. In particular if the mean curvature field of such a submanifold is parallel,

Email address: petrov@univer.kharkov.ua (E.V. Petrov)

then the Gauss map is harmonic (Propositions 4 and 5). Finally, in Section 5 we consider the case of hypersurfaces. We find out that the Gauss map of a cylindrical hypersurface is harmonic if and only if the mean curvature is constant. (Corollaries 6 and 7). Moreover, a constant mean curvature hypersurface with the harmonic Gauss map is cylindrical (Theorem 8).

2. Preliminaries

Suppose M is an n-dimensional smooth manifold, $\Psi\colon M \to N$ is an immersion of M in some $(n+q)$-dimensional Lie group N with a left invariant metric. For some point p of M let $Y_1, \dots, Y_n$ and $Y_{n+1}, \dots, Y_{n+q}$ be orthonormal frames of tangent space $T_pM \subset T_pN$ and of normal space $N_pM \subset T_pN$ respectively. Also, by Y_a, $1 \leqslant a \leqslant n+q$, denote the corresponding left invariant fields on N. By $(\cdot)^T$ and $(\cdot)^\perp$ denote the orthogonal projections on T_pM and N_pM respectively.

Denote the left invariant metric on N (and also the corresponding inner product on its Lie algebra) by $\langle \cdot, \cdot \rangle$, the Riemannian connection of this metric by ∇, its curvature tensor by $\mathrm{R}(\cdot,\cdot)\cdot$, and the normal connection of the immersion $M \to N$ by $\nabla^\perp$.

For $1 \leqslant i, j \leqslant n$, $n+1 \leqslant \alpha \leqslant n+q$ by b^α_{ij} denote the coefficients of the second fundamental form of the immersion with respect to the chosen frame. By H denote the mean curvature field of the immersion.

Let Φ be the Gauss map of M:

$$\Phi\colon M \to G(n,q);\ \Phi(p) = dL_{p^{-1}}(T_pM). \tag{1}$$

Here $G(n,q)$ is the Grassmannian of n-dimensional subspaces in an $(n+q)$-dimensional vector space, a point p is identified with its image under the immersion, $dL_{p^{-1}}$ is the differential of the left translation.

If (M_1, g_1) and (M_2, g_2) are smooth Riemannian manifolds, then for any $\phi \in C^\infty(M_1, M_2)$ the energy of ϕ is

$$E(\phi) = \frac{1}{2} \int\limits_{M_1} \sum_{1 \leqslant i \leqslant m} g_2(d\phi(E_i), d\phi(E_i)) dV_M,$$

where m is the dimension of M_1, $E_1, \dots, E_m$ is the orthonormal frame on M_1, dV_M is the volume form of g_1. The critical points of the functional $\phi \mapsto E(\phi)$ are called harmonic maps from M_1 to M_2. We say that a map is harmonic at some point if the corresponding Euler-Lagrange equations are satisfied at this point (i.e., the so-called tension field vanishes, see [2]).

The following was proved in [4].

Theorem 1 ([4, Theorem 1]). *The Gauss map Φ is harmonic at p if and only if*

$$\begin{aligned}\langle [nH, Y_j], Y_\alpha \rangle &+ \sum_{1 \leqslant i \leqslant n} \Big(\langle \mathrm{R}(Y_j, Y_i) Y_i, Y_\alpha \rangle - \langle \nabla_{(\nabla_{Y_i} Y_i)} Y_j, Y_\alpha \rangle - \langle \big(\nabla_{Y_i} Y_j\big)^T, (\nabla_{Y_i} Y_\alpha)^T \rangle \\ &+ \langle \big(\nabla_{Y_i} Y_j\big)^\perp, (\nabla_{Y_i} Y_\alpha)^\perp \rangle \Big) + 2 \sum_{1 \leqslant i,k \leqslant n} b^\alpha_{ik} \langle \nabla_{Y_i} Y_k, Y_j \rangle + 2 \sum_{1 \leqslant i \leqslant n, n+1 \leqslant \gamma \leqslant n+q} b^\gamma_{ij} \langle \nabla_{Y_i} Y_\gamma, Y_\alpha \rangle = 0\end{aligned} \tag{2}$$

for $1 \leqslant j \leqslant n$, $n+1 \leqslant \alpha \leqslant n+q$.

Also, it was shown in [4] that the terms in (2) not including H and the coefficients of the second fundamental form can be rewritten as

$$\sum_{1\leqslant i\leqslant n}\left\langle [Y_j,\nabla_{Y_i}Y_i]+[Y_i,[Y_j,Y_i]]+2\nabla_{Y_i}\left(\left([Y_i,Y_j]\right)^T-\left(\nabla_{Y_j}Y_i\right)^{\perp}\right),Y_\alpha\right\rangle. \tag{3}$$

Let now N be the $(2m+1)$-dimensional Heisenberg group ([6]). It is $\mathbb{R}^{2m+1}$ with coordinates $x^1,\dots,x^m,y^1,\dots,y^m,z$. Left-invariant vector fields

$$K_1=\frac{\partial}{\partial x^1},\dots,K_m=\frac{\partial}{\partial x^m},\ L_1=\frac{\partial}{\partial y^1}+x^1\frac{\partial}{\partial z},\dots,L_m=\frac{\partial}{\partial y^m}+x^m\frac{\partial}{\partial z},\ Z=\frac{\partial}{\partial z} \tag{4}$$

form a basis. The Lie algebra structure is defined by

$$[K_i,L_j]=\delta_{ij}Z,\ [K_i,K_j]=[L_i,L_j]=[K_i,Z]=[L_i,Z]=0 \tag{5}$$

for $1\leqslant i,j\leqslant m$, where δ_{ij} is the Kronecker symbol. The center $\mathcal{Z}$ of the Lie algebra is one-dimensional space generated by Z. Consider a left invariant metric on N such that (4) is orthonormal. Denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{Z}$. Note that $0\neq[\mathcal{V},\mathcal{V}]\subset\mathcal{Z}$. Lie algebras (and corresponding Lie groups) with this property are called 2-step nilpotent (see [6]). For each $Z^*\in\mathcal{Z}$ define the linear operator $J(Z^*)$ on $\mathcal{V}$ by $\langle J(Z^*)X,Y\rangle=\langle[X,Y],Z^*\rangle$. Note that $J(Z)^2=-\mathrm{Id}$.

The left invariant Riemannian (Levi-Civita) connection on N corresponding to the chosen metric is defined by (see [6])

$$\begin{aligned}\nabla_XY&=\tfrac{1}{2}[X,Y];\\ \nabla_XZ^*&=\nabla_{Z^*}X=-\tfrac{1}{2}J(Z^*)X;\\ \nabla_{Z^*}Z^{**}&=0.\end{aligned} \tag{6}$$

The curvature tensor is defined by

$$\begin{aligned}\mathrm{R}(X,Y)X^*&=\frac{1}{2}J([X,Y])X^*-\frac{1}{4}J([Y,X^*])X+\frac{1}{4}J([X,X^*])Y;\\ \mathrm{R}(X,Z^*)Y&=-\frac{1}{4}[X,J(Z^*)Y];\\ \mathrm{R}(X,Y)Z^*&=-\frac{1}{4}[X,J(Z^*)Y]+\frac{1}{4}[Y,J(Z^*)X];\\ \mathrm{R}(X,Z^*)Z^{**}&=-\frac{1}{4}J(Z^*)J(Z^{**})X;\end{aligned} \tag{7}$$

in particular, $R(X,Y)X=\frac{3}{4}J([X,Y])X$. Here $X,X^*,Y\in\mathcal{V},Z^*,Z^{**}\in\mathcal{Z}$.

3. The harmonicity criterion

So, let N be the $(2m+1)$–dimensional Heisenberg group and M be a submanifold immersed in N. Choose an orthonormal frame at p such that

$$Y_a=X_a,\ 1\leqslant a\leqslant n-1,\ n+2\leqslant a\leqslant n+q,\ Y_n=X_n-|X_{n+1}|Z,\ Y_{n+1}=X_{n+1}+|X_n|Z, \tag{8}$$

where $X_1,\dots,X_{n-1},X_{n+2},\dots,X_{n+q}$ is an orthonormal system in $\mathcal{V}$, $X_n\in\mathcal{V}$ and $X_{n+1}\in\mathcal{V}$ are orthogonal to $X_1,\dots,X_{n-1},X_{n+2},\dots,X_{n+q}$, $X_{n+1}=\lambda X_n$, where $\lambda\geqslant 0$, $|X_n|^2+|X_{n+1}|^2=1$.

Proposition 2. *The Gauss map of an immersed submanifold M in the Heisenberg group N with the chosen metric is harmonic at a point p if and only if, in the notation introduced above,*

$$\begin{aligned} &\langle [nH, Y_j], Y_\alpha\rangle - \left(|X_{n+1}|^2 + \frac{1}{2}\right)\langle \left(J(Z)X_j\right)^T, (J(Z)X_\alpha)^T\rangle \\ &-|X_{n+1}|^2\langle J(Z)X_j, X_n\rangle\langle J(Z)X_\alpha, X_n\rangle \\ &+2|X_{n+1}|\sum_{1\leqslant i\leqslant n} b^\alpha_{in}\langle J(Z)X_i, X_j\rangle - |X_n|\sum_{1\leqslant i\leqslant n} b^{n+1}_{ij}\langle J(Z)X_i, X_\alpha\rangle \\ &+|X_{n+1}|\sum_{n+1\leqslant\gamma\leqslant n+q} b^\gamma_{nj}\langle J(Z)X_\gamma, X_\alpha\rangle = 0 \end{aligned} \tag{9}$$

for $1 \leqslant j \leqslant n-1 \;\; n+2 \leqslant \alpha \leqslant n+q$;

$$\begin{aligned} &\langle [nH, Y_n], Y_\alpha\rangle - \left(|X_{n+1}|^2 + \frac{1}{2}\right)\langle (J(Z)X_n)^T, (J(Z)X_\alpha)^T\rangle + 2|X_{n+1}|\sum_{1\leqslant i\leqslant n} b^\alpha_{in}\langle J(Z)X_i, X_n\rangle \\ &-|X_n|\sum_{1\leqslant i\leqslant n} b^{n+1}_{in}\langle J(Z)X_i, X_\alpha\rangle + |X_{n+1}|\sum_{n+1\leqslant\gamma\leqslant n+q} b^\gamma_{nn}\langle J(Z)X_\gamma, X_\alpha\rangle = 0 \end{aligned} \tag{10}$$

for $n+2 \leqslant \alpha \leqslant n+q$;

$$\begin{aligned} &\langle [nH, Y_j], Y_{n+1}\rangle - \left(|X_{n+1}|^2 + \frac{1}{2}\right)\langle \left(J(Z)X_j\right)^T, (J(Z)X_{n+1})^T\rangle \\ &+2|X_{n+1}|\sum_{1\leqslant i\leqslant n} b^{n+1}_{in}\langle J(Z)X_i, X_j\rangle \\ &+|X_n|\sum_{1\leqslant i\leqslant n, n+2\leqslant\gamma\leqslant n+q} b^\gamma_{ij}\langle J(Z)X_i, X_\gamma\rangle + |X_{n+1}|\sum_{n+2\leqslant\gamma\leqslant n+q} b^\gamma_{nj}\langle J(Z)X_\gamma, X_{n+1}\rangle = 0 \end{aligned} \tag{11}$$

for $1 \leqslant j \leqslant n-1$*; and*

$$\begin{aligned} &\langle [nH, Y_n], Y_{n+1}\rangle - \left(|X_{n+1}|^2 + \frac{1}{2}\right)\langle (J(Z)X_n)^T, (J(Z)X_{n+1})^T\rangle \\ &+\frac{1}{2}\,|X_{n+1}||X_n|\left(2|X_{n+1}|^2 - n + 1 + \sum_{1\leqslant i,k\leqslant n}|[X_i, X_k]|^2\right) \\ &+2|X_{n+1}|\sum_{1\leqslant i\leqslant n} b^{n+1}_{in}\langle J(Z)X_i, X_n\rangle + |X_n|\sum_{1\leqslant i\leqslant n, n+2\leqslant\gamma\leqslant n+q} b^\gamma_{in}\langle J(Z)X_i, X_\gamma\rangle \\ &+|X_{n+1}|\sum_{n+2\leqslant\gamma\leqslant n+q} b^\gamma_{nn}\langle J(Z)X_\gamma, X_{n+1}\rangle = 0. \end{aligned} \tag{12}$$

Proof. The terms that include the coefficients of the second fundamental form are obtained directly from (2) and the expressions (6). Indeed, for $1 \leqslant j \leqslant n$ and $n+1 \leqslant \alpha \leqslant n+q$

$$\begin{aligned} &2\sum_{1\leqslant i,k\leqslant n} b^\alpha_{ik}\langle \nabla_{Y_i} Y_k, Y_j\rangle = \sum_{1\leqslant i,k\leqslant n} b^\alpha_{ik}\langle [X_i, X_k], Y_j\rangle \\ &+2|X_{n+1}|\sum_{1\leqslant i\leqslant n} b^\alpha_{in}\langle J(Z)X_i, X_j\rangle = 2|X_{n+1}|\sum_{1\leqslant i\leqslant n} b^\alpha_{in}\langle J(Z)X_i, X_j\rangle, \end{aligned}$$

because the second fundamental form is symmetric and the Lie bracket is skew-symmetric. For $1 \leqslant j \leqslant n$ and $n+1 \leqslant \alpha \leqslant n+q$

$$2 \sum_{1\leqslant i\leqslant n, n+1\leqslant \gamma\leqslant n+q} b^{\gamma}_{ij} \langle \nabla_{Y_i} Y_\gamma, X_\alpha\rangle = -|X_n| \sum_{1\leqslant i\leqslant n} b^{n+1}_{ij} \langle J(Z)X_i, X_\alpha\rangle$$
$$+|X_{n+1}| \sum_{n+1\leqslant\gamma\leqslant n+q} b^{\gamma}_{nj} \langle J(Z)X_\gamma, X_\alpha\rangle.$$

Also, for $1 \leqslant j \leqslant n$

$$2 \sum_{1\leqslant i\leqslant n, n+1\leqslant \gamma\leqslant n+q} b^{\gamma}_{ij} \langle \nabla_{Y_i} Y_\gamma, Z\rangle = \sum_{1\leqslant i\leqslant n, n+1\leqslant \gamma\leqslant n+q} b^{\gamma}_{ij} \langle [X_i, X_\gamma], Z\rangle$$
$$= \sum_{1\leqslant i\leqslant n, n+1\leqslant \gamma\leqslant n+q} b^{\gamma}_{ij} \langle J(Z)X_i, X_\gamma\rangle.$$

In order to find other terms in (2) use (3) (note that for a 2-step nilpotent Lie algebra the second term in (3) vanishes). Denote by P_j for $1 \leqslant j \leqslant n$ the vector

$$\sum_{1\leqslant i\leqslant n} \left([X_j, \nabla_{Y_i} Y_i] + 2\nabla_{Y_i} \left([Y_i, X_j]^T - \left(\nabla_{X_j} Y_i\right)^\perp \right)\right)^\perp$$
$$= \Bigg([X_j, |X_{n+1}| J(Z)X_n] + 2 \sum_{1\leqslant i\leqslant n-1} \nabla_{X_i} \left([X_i, X_j]^T - \frac{1}{2}\, [X_j, X_i]^\perp \right)$$
$$+2\nabla_{X_n - |X_{n+1}|Z} \left([X_n, X_j]^T - \frac{1}{2} \left([X_j, X_n] + |X_{n+1}| J(Z)X_j \right)^\perp \right)\Bigg)^\perp .$$

Note that $Y^T = \sum_{1\leqslant i\leqslant n} \langle Y, Y_i\rangle Y_i$, $Y^\perp = \sum_{n+1\leqslant\gamma\leqslant n+q} \langle Y, Y_\gamma\rangle Y_\gamma$. Using the frame in (8) obtain for $1 \leqslant i, j \leqslant n$

$$\begin{aligned} &[X_i, X_j]^T - \frac{1}{2}\, [X_j, X_i]^\perp = \frac{1}{2}\, [X_i, X_j] + \frac{1}{2}\, [X_i, X_j]^T \\ &= \frac{1}{2}\, [X_i, X_j] - \frac{1}{2}\, |X_{n+1}| \langle [X_i, X_j], Z\rangle (X_n - |X_{n+1}| Z). \end{aligned} \tag{13}$$

For $1 \leqslant j \leqslant n$

$$\left(J(Z)X_j\right)^\perp = \langle J(Z)X_j, X_{n+1}\rangle (X_{n+1} + |X_n| Z) + \sum_{n+2\leqslant\gamma\leqslant n+q} \langle J(Z)X_j, X_\gamma\rangle X_\gamma. \tag{14}$$

Consequently,

$$P_j = \Bigg(-|X_{n+1}| [J(Z)X_n, X_j] + \sum_{1\leqslant i\leqslant n} \nabla_{X_i} [X_i, X_j]$$
$$-|X_{n+1}| \sum_{1\leqslant i\leqslant n} \langle [X_i, X_j], Z\rangle \nabla_{X_i} (X_n - |X_{n+1}| Z) + |X_{n+1}|^2 \langle [X_n, X_j], Z\rangle \nabla_Z X_n$$
$$-|X_n||X_{n+1}| \langle J(Z)X_j, X_{n+1}\rangle \nabla_{X_n} Z - |X_{n+1}| \sum_{n+1\leqslant\gamma\leqslant n+q} \langle J(Z)X_j, X_\gamma\rangle \nabla_{X_n - |X_{n+1}|Z} X_\gamma \Bigg)^\perp .$$

Taking the covariant derivatives obtain

$$P_j = \Bigg(-|X_{n+1}|[J(Z)X_n, X_j] - \frac{1}{2} \sum_{1\leqslant i\leqslant n} J([X_i, X_j])X_i$$
$$-\frac{1}{2}|X_{n+1}| \sum_{1\leqslant i\leqslant n} \langle [X_i, X_j], Z\rangle [X_i, X_n] - \frac{1}{2}|X_{n+1}|^2 \sum_{1\leqslant i\leqslant n} \langle [X_i, X_j], Z\rangle J(Z)X_i$$
$$-\frac{1}{2}|X_{n+1}|^2 \langle [X_n, X_j], Z\rangle J(Z)X_n + \frac{1}{2}|X_n||X_{n+1}|\langle J(Z)X_j, X_{n+1}\rangle J(Z)X_n$$
$$-\frac{1}{2}|X_{n+1}| \sum_{n+1\leqslant\gamma\leqslant n+q} \langle J(Z)X_j, X_\gamma\rangle [X_n, X_\gamma] - \frac{1}{2}|X_{n+1}|^2 \sum_{n+1\leqslant\gamma\leqslant n+q} \langle J(Z)X_j, X_\gamma\rangle J(Z)X_\gamma \Bigg)^{\perp}.$$

Note that

$$|X_{n+1}|^2 \langle [X_n, X_j], Z\rangle = -|X_{n+1}|^2 \langle J(Z)X_j, X_n\rangle = -|X_n||X_{n+1}|\langle J(Z)X_j, X_{n+1}\rangle,$$

because X_n and X_{n+1} are proportional. Therefore,

$$P_j = \Bigg(-|X_{n+1}|[J(Z)X_n, X_j] - \frac{1}{2} \sum_{1\leqslant i\leqslant n} J([X_i, X_j])X_i$$
$$+\frac{1}{2}|X_{n+1}| \sum_{1\leqslant a\leqslant n+q} \langle J(Z)X_j, X_a\rangle [X_a, X_n] + |X_{n+1}|^2 \sum_{1\leqslant i\leqslant n} \langle J(Z)X_j, X_i\rangle J(Z)X_i$$
$$+|X_{n+1}|^2 \langle J(Z)X_j, X_n\rangle J(Z)X_n - \frac{1}{2}|X_{n+1}|^2 \sum_{1\leqslant a\leqslant n+q} \langle J(Z)X_j, X_a\rangle J(Z)X_a \Bigg)^{\perp}.$$

Each $X \in \mathcal{V}$ has a decomposition $X = \sum_{1\leqslant a\leqslant n+q} \langle X, X_a\rangle X_a$, thus,

$$P_j = \Bigg(-|X_{n+1}|[J(Z)X_n, X_j] + |X_{n+1}|^2 \langle J(Z)X_j, X_n\rangle J(Z)X_n + \frac{1}{2}|X_{n+1}|[J(Z)X_j, X_n]$$
$$+|X_{n+1}|^2 \sum_{1\leqslant i\leqslant n} \langle J(Z)X_j, X_i\rangle J(Z)X_i - \frac{1}{2}|X_{n+1}|^2 J(Z)^2 X_j - \frac{1}{2} \sum_{1\leqslant i\leqslant n} J([X_i, X_j])X_i \Bigg)^{\perp}.$$

Note that for all $1 \leqslant i, j \leqslant n$

$$[J(Z)X_i, X_j] = \langle [J(Z)X_i, X_j], Z\rangle Z = -\langle J(Z)X_i, J(Z)X_j\rangle Z$$
$$= -\langle [X_i, J(Z)X_j], Z\rangle Z = \langle [J(Z)X_j, X_i], Z\rangle Z = [J(Z)X_j, X_i].$$

This and the equation $J(Z)^2 = -\operatorname{Id}$ yield that

$$P_j = \Bigg(-\frac{1}{2}|X_{n+1}|[J(Z)X_n, X_j] + |X_{n+1}|^2 \langle J(Z)X_j, X_n\rangle J(Z)X_n$$
$$+\frac{1}{2}|X_{n+1}|^2 X_j + |X_{n+1}|^2 \sum_{1\leqslant i\leqslant n} \langle J(Z)X_j, X_i\rangle J(Z)X_i - \frac{1}{2} \sum_{1\leqslant i\leqslant n} J([X_i, X_j])X_i \Bigg)^{\perp}.$$

For $n+2 \leqslant \alpha \leqslant n+q$

$$\langle P_j, X_\alpha\rangle = |X_{n+1}|^2 \langle J(Z)X_j, X_n\rangle\langle J(Z)X_n, X_\alpha\rangle$$
$$+|X_{n+1}|^2 \sum_{1\leqslant i\leqslant n} \langle J(Z)X_j, X_i\rangle\langle J(Z)X_i, X_\alpha\rangle - \frac{1}{2} \sum_{1\leqslant i\leqslant n} \langle J([X_i, X_j])X_i, X_\alpha\rangle.$$

For $1 \leqslant j \leqslant n$ the inner product $\langle X_j, X_{n+1}\rangle$ equals $|X_n||X_{n+1}|\delta_{jn}$. The operator $J(Z)$ is skew-symmetric, thus $\langle J(Z)X_n, X_{n+1}\rangle = 0$. Hence,

$$\langle P_j, X_{n+1} + |X_n|Z\rangle = -\frac{1}{2}\,|X_{n+1}||X_n|\langle [J(Z)X_n, X_j], Z\rangle + \frac{1}{2}\,|X_{n+1}|^3|X_n|\delta_{jn}$$
$$+|X_{n+1}|^2 \sum_{1\leqslant i\leqslant n} \langle J(Z)X_j, X_i\rangle\langle J(Z)X_i, X_{n+1}\rangle - \frac{1}{2} \sum_{1\leqslant i\leqslant n} \langle J([X_i, X_j])X_i, X_{n+1}\rangle.$$

Also, for all $n+1 \leqslant \alpha \leqslant n+q$

$$\sum_{1\leqslant i\leqslant n} \langle J([X_i, X_j])X_i, X_\alpha\rangle = \sum_{1\leqslant i\leqslant n} \langle [X_i, X_j], [X_i, X_\alpha]\rangle = \sum_{1\leqslant i\leqslant n} \langle [X_i, X_j], Z\rangle\langle [X_i, X_\alpha], Z\rangle$$
$$= \sum_{1\leqslant i\leqslant n} \langle J(Z)X_j, X_i\rangle\langle J(Z)X_\alpha, X_i\rangle = \langle \left(J(Z)X_j\right)^T, (J(Z)X_\alpha)^T\rangle.$$

Note also that

$$\langle [J(Z)X_n, X_j], Z\rangle = \langle J(Z)^2 X_n, X_j\rangle = -\langle X_n, X_j\rangle = -|X_n|^2\delta_{jn}.$$

It follows that

$$\begin{aligned} P_j = \Bigg(&\frac{1}{2}\,|X_{n+1}||X_n|(|X_{n+1}|^2 + |X_n|^2)\delta_{jn} \\ &-\left(|X_{n+1}|^2 + \frac{1}{2}\right)\langle \left(J(Z)X_j\right)^T, (J(Z)X_{n+1})^T\rangle\Bigg)(X_{n+1} + |X_n|Z) \\ &+ \sum_{n+2\leqslant\alpha\leqslant n+q} \Bigg(-|X_{n+1}|^2\langle J(Z)X_j, X_n\rangle\langle J(Z)X_\alpha, X_n\rangle \\ &-\left(|X_{n+1}|^2 + \frac{1}{2}\right)\langle \left(J(Z)X_j\right)^T, (J(Z)X_\alpha)^T\rangle\Bigg) X_\alpha. \end{aligned} \tag{15}$$

This implies (9) and (11). Now consider the expressions in (2) for $Y_n = X_n - |X_{n+1}|Z$. Denote

$$Q = \sum_{1\leqslant i\leqslant n} \left([Y_n, \nabla_{Y_i} Y_i] + 2\nabla_{Y_i}\left(([Y_i, Y_n])^T - (\nabla_{Y_n} Y_i)^\perp\right)\right)^\perp = P_n + 2|X_{n+1}| \sum_{1\leqslant i\leqslant n} \left(\nabla_{Y_i}\,(\nabla_Z Y_i)^\perp\right)^\perp.$$

It follows from the expression for Y_n that

$$Q = P_n + |X_{n+1}|\left(-\sum_{1\leqslant i\leqslant n} \nabla_{X_i}\,(J(Z)X_i)^\perp + |X_{n+1}|\nabla_Z\,(J(Z)X_n)^\perp\right)^\perp.$$

Now use (14).

$$Q = P_n + |X_{n+1}| \Bigg(- \sum_{1\leqslant i\leqslant n, n+1\leqslant \gamma\leqslant n+q} \langle J(Z)X_i, X_\gamma\rangle \nabla_{X_i} X_\gamma$$

$$- |X_n| \sum_{1\leqslant i\leqslant n} \langle J(Z)X_i, X_{n+1}\rangle \nabla_{X_i} Z + |X_{n+1}| \sum_{n+1\leqslant \gamma\leqslant n+q} \langle J(Z)X_n, X_\gamma\rangle \nabla_Z X_\gamma \Bigg)^{\perp}.$$

It is easy to see that $|X_n|\langle J(Z)X_a, X_{n+1}\rangle = |X_{n+1}|\langle J(Z)X_a, X_n\rangle$ for $1 \leqslant a \leqslant n+q$, thus,

$$\begin{aligned} Q = P_n - \frac{1}{2}|X_{n+1}| \Bigg(\sum_{1\leqslant i\leqslant n, n+1\leqslant\gamma\leqslant n+q} \langle J(Z)X_i, X_\gamma\rangle [X_i, X_\gamma] + |X_{n+1}| \sum_{1\leqslant a\leqslant n+q} \langle J(Z)X_n, X_a\rangle J(Z)X_a \Bigg)^{\perp} \\ = P_n + \frac{1}{2}\,|X_{n+1}| \Bigg(- \sum_{1\leqslant i\leqslant n, 1\leqslant a\leqslant n+q} \langle J(Z)X_i, X_a\rangle [X_i, X_a] \\ + \sum_{1\leqslant i,k\leqslant n} \langle J(Z)X_i, X_k\rangle [X_i, X_k] - |X_{n+1}| J(Z)^2 X_n \Bigg)^{\perp} \\ = P_n + \frac{1}{2}\,|X_{n+1}| \Bigg(- \sum_{1\leqslant i\leqslant n} [X_i, J(Z)X_i] + \sum_{1\leqslant i,k\leqslant n} \langle J(Z)X_i, X_k\rangle [X_i, X_k] + |X_{n+1}| X_n \Bigg)^{\perp}. \end{aligned}$$

Decompose Q with respect to the frame of the normal space.

$$Q = P_n + \frac{1}{2}\,|X_n||X_{n+1}| \Bigg(- \sum_{1\leqslant i\leqslant n} \langle [X_i, J(Z)X_i], Z\rangle$$

$$+ \sum_{1\leqslant i,k\leqslant n} \langle J(Z)X_i, X_k\rangle \langle [X_i, X_k], Z\rangle + |X_{n+1}|^2 \Bigg) (X_{n+1} + |X_n| Z).$$

The properties of $J(Z)$ and (8) yield

$$\sum_{1\leqslant i\leqslant n} \langle [X_i, J(Z)X_i], Z\rangle = - \sum_{1\leqslant i\leqslant n} \langle X_i, J(Z)^2 X_i\rangle = \sum_{1\leqslant i\leqslant n} \langle X_i, X_i\rangle = n - 1 + |X_n|^2.$$

Using also the fact that $|\langle [X_i, X_k], Z\rangle| = |[X_i, X_k]|$ and (11) rewrite

$$\begin{aligned} Q = \Bigg(\frac{1}{2}\,|X_{n+1}||X_n| \Bigg(2|X_{n+1}|^2 - n + 1 + \sum_{1\leqslant i,k\leqslant n} |[X_i, X_k]|^2 \Bigg) \\ - \left(|X_{n+1}|^2 + \frac{1}{2} \right) \langle (J(Z)X_n)^T, (J(Z)X_{n+1})^T \rangle \Bigg) (X_{n+1} + |X_n| Z) \\ + \sum_{n+2\leqslant\alpha\leqslant n+q} \left(- \left(|X_{n+1}|^2 + \frac{1}{2} \right) \langle (J(Z)X_n)^T, (J(Z)X_\alpha)^T \rangle \right) X_\alpha. \end{aligned}$$

This implies (10) and (12). □

Note that for $q = 1$ and for the special choice of the frame (11) and (12) imply the conditions of Proposition 7 in [5] (actually, our Z is opposite to Z from that paper).

Consider some examples using the conditions in Proposition 2. Let $1 \leqslant l \leqslant k \leqslant m$, $l + k = n$, $2 \leqslant n \leqslant 2m$. Consider n-dimensional horizontal distribution F in N generated by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^l}$. It is obviously integrable and generates some foliation $\mathcal{F}$.

Proposition 3. *Each leaf of $\mathcal{F}$ is minimal. There exist leafs with the nonharmonic Gauss map.*

Proof. Consider the orthonormal frame

$$F_1 = K_1, \ldots, F_k = K_k, G_1 = \frac{L_1 - x^1 Z}{\sqrt{1 + (x^1)^2}}, \ldots, G_l = \frac{L_l - x^l Z}{\sqrt{1 + (x^l)^2}}. \tag{16}$$

It follows from (6) that $\nabla_{F_i} F_j = 0$ for $1 \leqslant i, j \leqslant k$. Let $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$. If $i \neq j$, then (6) implies

$$\nabla_{F_i} G_j = \nabla_{G_j} F_i = \frac{\frac{1}{2}\, x^j L_i}{\sqrt{1 + (x^j)^2}}.$$

For $1 \leqslant i \leqslant l$ the derivative $\nabla_{F_i} G_i$ equals

$$\frac{-Z\left(1 + \left(x^i\right)^2\right) - \left(L_i - x^i Z\right) x^i}{\left(1 + (x^i)^2\right)^{\frac{3}{2}}} + \frac{\frac{1}{2}\left(Z + x^i L_i\right)}{\sqrt{1 + (x^i)^2}} = \frac{\frac{1}{2}\left(x^i L_i + Z\right)\left(\left(x^i\right)^2 - 1\right)}{\left(1 + (x^i)^2\right)^{\frac{3}{2}}}.$$

For $1 \leqslant i, j \leqslant l$

$$\nabla_{G_i} G_j = \frac{-\frac{1}{2}\left(x^i K_j + x^j K_i\right)}{\sqrt{1 + (x^i)^2}\sqrt{1 + (x^j)^2}}.$$

The frame is orthonormal, thus,

$$H = \frac{1}{n}\left(\sum_{1 \leqslant i \leqslant k} (\nabla_{F_i} F_i)^{\perp} + \sum_{1 \leqslant j \leqslant l} \left(\nabla_{G_j} G_j\right)^{\perp}\right) = -\frac{1}{n}\sum_{1 \leqslant j \leqslant l}\left(\frac{x^j K_j}{1 + (x^j)^2}\right)^{\perp} = 0,$$

because all K_j are tangent. Therefore each leaf is minimal. Note also that $b_{ii}^{\alpha} = 0$ for all $1 \leqslant i \leqslant n$, $n + 1 \leqslant \alpha \leqslant n + q$.

Consider (12) at $p = (0, \ldots, x_0^l, \ldots, 0)$, where $x_0^l > 0$. The local orthonormal frame is formed by

$$X_1 = K_1, \ldots, X_k = K_k, X_{k+1} = L_1, \ldots, X_{n-1} = L_{l-1},$$
$$Y_n = X_n - |X_{n+1}| Z = \frac{L_l - x_0^l Z}{\sqrt{1 + \left(x_0^l\right)^2}}, \; Y_{n+1} = X_{n+1} + |X_n| Z = \frac{x_0^l L_l + Z}{\sqrt{1 + \left(x_0^l\right)^2}},$$

and the other normal vectors are chosen from K_i L_j. Then $|X_n| = \frac{1}{\sqrt{1+\left(x_0^l\right)^2}}$ and $|X_{n+1}| = \frac{x_0^l}{\sqrt{1+\left(x_0^l\right)^2}}$.

It follows from $l \leqslant k$ that

$$\sum_{1\leqslant i,k\leqslant n} \|[X_i, X_k]\|^2 = 2\left(l - 1 + \frac{1}{1+\left(x_0^l\right)^2}\right);$$

$$\sum_{1\leqslant i\leqslant n, n+2\leqslant \gamma\leqslant n+q} b_{in}^{\gamma}\langle J(Z)X_i, X_\gamma\rangle = \sum_{l+1\leqslant i\leqslant k, n+2\leqslant\gamma\leqslant n+q} \langle \frac{\frac{1}{2}\, x_0^l L_i}{\sqrt{1+\left(x_0^l\right)^2}}, X_\gamma\rangle\langle L_i, X_\gamma\rangle$$

$$= \frac{1}{2}\,\frac{x_0^l}{\sqrt{1+\left(x_0^l\right)^2}} \sum_{l+1\leqslant i\leqslant k, n+2\leqslant\gamma\leqslant n+q} \langle L_i, X_\gamma\rangle^2 = \frac{1}{2}\,\frac{x_0^l}{\sqrt{1+\left(x_0^l\right)^2}}(k-l);$$

$$\sum_{1\leqslant i\leqslant n} b_{in}^{n+1}\langle J(Z)X_i, X_n\rangle = b_{ln}^{n+1}\langle J(Z)X_l, X_n\rangle = \frac{\frac{1}{2}\left(\left(x_0^l\right)^2 - 1\right)}{1+\left(x_0^l\right)^2}\langle L_l, \frac{L_l}{\sqrt{1+\left(x_0^l\right)^2}}\rangle = \frac{1}{2}\,\frac{\left(x_0^l\right)^2 - 1}{\left(1+\left(x_0^l\right)^2\right)^{\frac{3}{2}}}.$$

So, (12) takes the form

$$-\left(\frac{\left(x_0^l\right)^2}{1+\left(x_0^l\right)^2} + \frac{1}{2}\right)\langle \frac{-K_l}{\sqrt{1+\left(x_0^l\right)^2}}, \frac{-x_0^l K_l}{\sqrt{1+\left(x_0^l\right)^2}}\rangle + \frac{1}{2}\,\frac{x_0^l}{1+\left(x_0^l\right)^2}\left(2\frac{\left(x_0^l\right)^2}{1+\left(x_0^l\right)^2} - n + 1\right.$$

$$\left.+2\left(l - 1 + \frac{1}{1+\left(x_0^l\right)^2}\right)\right) + \frac{x_0^l}{\sqrt{1+\left(x_0^l\right)^2}}\,\frac{\left(x_0^l\right)^2 - 1}{\left(1+\left(x_0^l\right)^2\right)^{\frac{3}{2}}} + \frac{1}{2}\,\frac{1}{\sqrt{1+\left(x_0^l\right)^2}}\,\frac{x_0^l}{\sqrt{1+\left(x_0^l\right)^2}}(k-l)$$

$$= \frac{x_0^l}{\left(1+\left(x_0^l\right)^2\right)^2}\left(\frac{1}{2}\left(1+\left(x_0^l\right)^2\right)(-n+l-2+k) + \left(x_0^l\right)^2\right) = -\frac{x_0^l}{\left(1+\left(x_0^l\right)^2\right)^2},$$

because $n = k + l$, and therefore does not vanish, thus the Gauss map is not harmonic. □

4. Cylindrical submanifolds

We call a submanifold M cylindrical if its tangent space at each point contains the value of the field Z. Integral trajectories of Z are $z = t$, therefore if M is complete, then $M = M_1 \times \mathbb{R}$, where M_1 is a smooth submanifold in the subspace $z = 0$. Consider a natural Euclidean metric on this $(2m)$-dimensional subspace with an orthonormal basis formed by $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^j}$ for $1 \leqslant i, j \leqslant m$. Denote this space by E^{2m}.

Proposition 4. *Let M be cylindrical. Then H is parallel if and only if the Gauss map of M is harmonic and $(J(Z)H)^\perp = 0$.*

Here $J(Z)H$ is the image of the value of the vector field H under the operator $J(Z)$ at each point (H is normal, hence is orthogonal to Z). In particular, if H is parallel, then the Gauss map is harmonic. It follows from Proposition 3 that this is not true in general for submanifolds in the Heisenberg group.

Proposition 5. *Let M be cylindrical and complete. Then*

(i). $M = M_1 \times \mathbb{R}$ is minimal in N if and only if M_1 is minimal in E^{2m};

(ii). H is parallel if and only if the mean curvature field H_1 of M_1 in E^{2m} is parallel and $(J(Z)H)^\perp = 0$;

(iii). the Gauss map of M is harmonic if and only if H_1 is parallel.

Proof of Propositions 4 and 5. Let us prove Proposition 5 first. Let

$$F_k = \sum_{1\leqslant i\leqslant m} \left(A_k^i \frac{\partial}{\partial x^i} + B_k^i \frac{\partial}{\partial y^i} \right)$$

for $1 \leqslant k \leqslant n-1$ form a local orthonormal (with respect to the metric induced by the immersion in E^{2m}) frame of the tangent bundle of M_1 on some neighborhood U_1 of the given point. Here A_i^k, B_i^k are functions on U_1. Then the tangent bundle of M on $U_1 \times \mathbb{R}$ has a local orthonormal frame $\{E_1, \ldots, E_{n-1}, Z\}$, where

$$E_k = \sum_{1\leqslant i\leqslant m} \left(A_k^i\ K_i + B_k^i\ L_i \right).$$

After the covariant differentiation obtain

$$\sum_{1\leqslant k\leqslant n-1} \nabla_{E_k} E_k = \sum_{1\leqslant k\leqslant n-1,\, 1\leqslant i\leqslant m} \left(E_k(A_k^i)\ K_i + E_k(B_k^i)\ L_i \right),$$

because $\nabla_{K_i} K_j = \nabla_{L_i} L_j = \nabla_{K_i} L_i + \nabla_{L_i} K_i = 0$ for $1 \leqslant i, j \leqslant m$ and $\nabla_{K_i} L_j = 0$ for $i \neq j$ due to (6) and (5). Here $X(f)$ is the derivative of f with respect to X. The functions A_i^k and B_i^k are independent of z. It follows that

$$H = \sum_{1\leqslant k\leqslant n-1,\, 1\leqslant i\leqslant m} \left(F_k(A_k^j) \frac{\partial}{\partial x^j} + F_k(B_k^i) \frac{\partial}{\partial y^i} \right)^\perp, \tag{17}$$

because $\nabla_Z Z = 0$. This expression coincides with the expression of H_1. Then the claim *(i)* follows. Also, (17) implies $\nabla_Z^\perp H = -\frac{1}{2}\ (J(Z)H)^\perp$ because the normal space at each point is contained in $\mathcal{V}$, thus (6) can be used. And hence *(ii)* follows.

The submanifold M is cylindrical, thus $X_n = 0$, $|X_{n+1}| = 1$. The vector field $-Z$ is the extension of $Y_n = -Z$ on a neighborhood of each point p. This and (6) imply that for $n+1 \leqslant \alpha \leqslant n+q$ coefficients b_{nn}^α vanish and for $1 \leqslant i \leqslant n-1$ at p

$$b_{in}^\alpha = \langle \nabla_{X_i}(-Z), X_\alpha \rangle = \frac{1}{2}\ \langle J(Z)X_i, X_\alpha \rangle.$$

The equations (10) and (12) give the condition $\langle [nH, -Z], X_\alpha \rangle = 0$ for $n+1 \leqslant \alpha \leqslant n+q$, i.e., $[H, Z]^\perp = 0$. Compute the Lie bracket and obtain $\nabla_Z^\perp H = (\nabla_H Z)^\perp = -\frac{1}{2}\ (J(Z)H)^\perp$. The above argument implies that this is always true. The expressions in (9) and (11) for $1 \leqslant j \leqslant n-1$,

$n+1 \leqslant \alpha \leqslant n+q$ take the form

$$\langle [nH, X_j], X_\alpha\rangle - \frac{3}{2}\,\langle \left(J(Z)X_j\right)^T, (J(Z)X_\alpha)^T\rangle + \sum_{1\leqslant i\leqslant n} \langle J(Z)X_i, X_\alpha\rangle\langle J(Z)X_i, X_j\rangle$$

$$+\frac{1}{2}\sum_{n+1\leqslant\gamma\leqslant n+q}\langle J(Z)X_j, X_\gamma\rangle\langle J(Z)X_\gamma, X_\alpha\rangle = -n\langle\nabla^\perp_{X_j}H, X_\alpha\rangle - \frac{3}{2}\,\langle\left(J(Z)X_j\right)^T, (J(Z)X_\alpha)^T\rangle$$

$$+\sum_{1\leqslant i\leqslant n}\langle J(Z)X_\alpha, X_i\rangle\langle J(Z)X_j, X_i\rangle - \frac{1}{2}\sum_{n+1\leqslant\gamma\leqslant n+q}\langle J(Z)X_j, X_\gamma\rangle\langle J(Z)X_\alpha, X_\gamma\rangle$$

$$= -n\langle\nabla^\perp_{X_j}H, X_\alpha\rangle - \frac{3}{2}\,\langle\left(J(Z)X_j\right)^T, (J(Z)X_\alpha)^T\rangle + \langle (J(Z)X_\alpha)^T, \left(J(Z)X_j\right)^T\rangle$$

$$-\frac{1}{2}\,\langle\left(J(Z)X_j\right)^\perp, (J(Z)X_\alpha)^\perp\rangle = -n\langle\nabla^\perp_{X_j}H, X_\alpha\rangle - \frac{1}{2}\,\langle J(Z)X_j, J(Z)X_\alpha\rangle$$

$$= -n\langle\nabla^\perp_{X_j}H, X_\alpha\rangle + \frac{1}{2}\,\langle J(Z)^2X_j, X_\alpha\rangle = -n\langle\nabla^\perp_{X_j}H, X_\alpha\rangle - \frac{1}{2}\,\langle X_j, X_\alpha\rangle = -n\langle\nabla^\perp_{X_j}H, X_\alpha\rangle,$$

because $J(Z)$ is skew-symmetric. Thus (9) and (11) are satisfied if and only if H_1 is parallel, and then *(iii)* follows.

Each point of cylindrical M has a neighborhood of the form $U = U_1\times(a, b)$, and the argument in the proofs of *(ii)* and *(iii)* in Proposition 5 are valid also for the proof of Proposition 4. □

Note that *(iii)* of Proposition 5 and the result of [1] imply that the Gauss map of M is harmonic if and only if the Gauss map of M_1 is harmonic.

5. Cylindrical hypersurfaces

Let M be a cylindrical hypersurface ($n = 2m, q = 1$). The normal space is one-dimensional. Hence from $\langle J(Z)H, H\rangle = 0$ it follows that at each point either $H = 0$ (thus $J(Z)H = 0$) or $J(Z)H$ is tangent. Therefore Propositions 4 and 5 and the remark after the proof of these propositions imply the following corollaries.

Corollary 6. *Let M be a cylindrical hypersurface. Then the Gauss map of M is harmonic if and only if M is of constant mean curvature.*

Corollary 7. *Let M be a complete cylindrical hypersurface. Then the following claims are equivalent:*

(i). the Gauss map of M is harmonic;
(ii). M is of constant mean curvature;
(iii). the Gauss map of M_1 in E^{2m} is harmonic;
(iv). M_1 is of constant mean curvature in E^{2m}.

Moreover, the following result is true.

Theorem 8. *Let M be a constant mean curvature hypersurface in N. If the Gauss map of M is harmonic, then M is cylindrical.*

Proof. Denote now by H the mean curvature function of M and by B the scalar second fundamental form (in the case of a non-orientable hypersurface these objects are defined only locally). By b_{ij} denote the coefficients of B: $b_{ij} = b_{ij}^{2m+1}$. Choose $X_1, \dots, X_{2m+1}$ at p such that

$$\begin{gathered} J(Z)X_i = X_{m+i},\ 1 \leqslant i \leqslant m-1; \\ J(Z)X_m = \frac{X_{2m}}{|X_{2m}|} \text{ if } X_{2m} \neq 0; \text{or} \frac{X_{2m+1}}{|X_{2m+1}|} \text{ if } X_{2m+1} \neq 0; \\ J(Z)X_{m+i} = -X_i,\ 1 \leqslant i \leqslant m-1; \\ J(Z)X_{2m} = -\,|X_{2m}|\,X_m;\ J(Z)X_{2m+1} = -\,|X_{2m+1}|\,X_m. \end{gathered} \tag{18}$$

The existence of such a frame follows from the existence of the basis of the vector space $\mathcal{V}$ such that the symplectic form $\langle J(Z)\cdot, \cdot\rangle$ has a canonical matrix.

Let the Gauss map of M be harmonic. In (2) replace $2mH$ by $2mH\eta$, where η is some unit normal vector field on a neighborhood of p extending Y_{2n+1}. The corresponding term at p takes the form

$$\begin{aligned} \langle [2mH\eta, Y_j], Y_{2m+1}\rangle &= \langle 2mH\nabla_{Y_{2m+1}} Y_j - Y_j(2mH)Y_{2m+1} - 2mH\nabla_{Y_j}\eta, Y_{2m+1}\rangle \\ &= 2mH\langle \nabla_{Y_{2m+1}} Y_j, Y_{2m+1}\rangle - Y_j(2mH), \end{aligned} \tag{19}$$

because $|\eta| = 1$. By the hypothesis H is constant, hence $Y_j(2mH) = 0$. Then for $1 \leqslant j \leqslant 2m-1$ the expressions in (11) become

$$\begin{gathered} 2mH\langle \nabla_{X_{2m+1}+|X_{2m}|Z} X_j, X_{2m+1} + |X_{2m}|Z\rangle - \left(|X_{2m+1}|^2 + \frac{1}{2}\right)\langle \big(J(Z)X_j\big)^T, (J(Z)X_{2m+1})^T\rangle \\ +2|X_{2m+1}| \sum_{1\leqslant i\leqslant 2m} b_{i\,2m}\langle J(Z)X_i, X_j\rangle = 0. \end{gathered}$$

The operator $J(Z)$ is skew-symmetric, hence $\langle J(Z)X_{2m+1}, X_{2m+1} + |X_{2m}|Z\rangle = 0$. Therefore,

$$\langle \big(J(Z)X_j\big)^T, (J(Z)X_{2m+1})^T\rangle = \langle J(Z)X_j, J(Z)X_{2m+1}\rangle = -\langle J(Z)^2X_j, X_{2m+1}\rangle = \langle X_j, X_{2m+1}\rangle = 0.$$

So, (11) takes the form

$$-2mH|X_{2m}|\langle J(Z)X_j, X_{2m+1}\rangle + 2|X_{2m+1}| \sum_{1\leqslant i\leqslant 2m} b_{i\,2m}\langle J(Z)X_i, X_j\rangle = 0. \tag{20}$$

The equation (19) imply for the constant H

$$\begin{aligned} \langle [2mH, Y_{2m}], Y_{2m+1}\rangle &= 2mH\langle \nabla_{X_{2m+1}+|X_{2m}|Z}\left(X_{2m} - |X_{2m+1}|Z\right), X_{2m+1} + |X_{2m}|Z\rangle \\ &= -2mH|X_{2m}|\langle J(Z)X_{2m}, X_{2m+1}\rangle + mH|X_{2m+1}|\langle J(Z)X_{2m+1}, X_{2m+1}\rangle = 0. \end{aligned}$$

The expressions in (12) turn to

$$\begin{gathered} -\left(|X_{2m+1}|^2 + \frac{1}{2}\right)\langle (J(Z)X_{2m})^T, (J(Z)X_{2m+1})^T\rangle + \frac{1}{2}\,|X_{2m+1}||X_{2m}|\Bigg(2|X_{2m+1}|^2 - 2m + 1 \\ + \sum_{1\leqslant i,k\leqslant 2m} |[X_i, X_k]|^2\Bigg) + 2|X_{2m+1}| \sum_{1\leqslant i\leqslant 2m} b_{i\,2m}\langle J(Z)X_i, X_{2m}\rangle = 0. \end{gathered}$$

Also, we have

$$\langle (J(Z)X_{2m})^T, (J(Z)X_{2m+1})^T\rangle = \langle X_{2m}, X_{2m+1}\rangle = |X_{2m}||X_{2m+1}|.$$

Now rewrite (12) in the form

$$\frac{1}{2}\, |X_{2m+1}||X_{2m}|\left(\sum_{1\leqslant i,k\leqslant 2m} |[X_i, X_k]|^2 - 2m\right) + 2|X_{2m+1}| \sum_{1\leqslant i\leqslant 2m} b_{i\,2m}\langle J(Z)X_i, X_{2m}\rangle = 0. \tag{21}$$

If the frame is as in (18), then for $1 \leqslant j \leqslant m-1$ the expressions (20) are of the form

$$-2|X_{2m+1}|b_{m+j\,2m} = 0. \tag{22}$$

For $j = m$ the conditions in (20) become

$$-|X_{2m}||X_{2m+1}|\,(2mH + 2b_{2m\,2m}) = 0. \tag{23}$$

Finally, if $m + 1 \leqslant j \leqslant 2m - 1$, then

$$2|X_{2m+1}|b_{j-m\,2m} = 0. \tag{24}$$

It follows from (18) that the only nonzero brackets of $X_1, \dots, X_{2m}$ are $[X_i, X_{m+i}] = -[X_{m+i}, X_i] = Z$ for $1 \leqslant i \leqslant m-1$ and $[X_m, X_{2m}] = -[X_{2m}, X_m] = |X_{2m}|Z$. Hence,

$$\sum_{1\leqslant i,k\leqslant 2m} |[X_i, X_k]|^2 = 2(m - 1 + |X_{2m}|^2) = 2(m - |X_{2m+1}|^2).$$

Thus (21) become

$$-|X_{2m+1}||X_{2m}|\left(|X_{2m+1}|^2 - 2b_{m\,2m}\right) = 0. \tag{25}$$

For each $p \in M$ denote $a(p) = |X_{2m+1}|$, $b(p) = |X_{2m}|$. Then a and b are smooth functions on M, $a^2 + b^2 = 1$. Denote by $T_1, \dots, T_{2m-1}$ vector fields on M that at each p are equal to $X_1, \dots, X_{2m-1}$, respectively. By T_{2m} denote a field that equals $J(Z)X_m$ at each $p \in M$. Then unit tangent vector fields of the hypersurface $F_1, \dots, F_{2m}$ and a unit normal vector field η are of the form

$$F_1 = T_1, \dots\ F_{2m-1} = T_{2m-1},\ F_{2m} = bT_{2m} - aZ,\ \eta = aT_{2m} + bZ.$$

Suppose that $a \neq 0$ and $b \neq 0$ at some point p. Then $ab \neq 0$ on some neighborhood U of p. It follows from (22)—(25) that on U

$$b_{j\,2m} = 0,\ j \neq m, 2m;\ b_{m\,2m} = \frac{a^2}{2};\ b_{2m\,2m} = -mH. \tag{26}$$

Note that

$$\eta = \frac{a}{b}\, F_{2m} + \left(\frac{a^2}{b} + b\right)Z = \frac{a}{b}\, F_{2m} + \frac{1}{b}\, Z.$$

Therefore for $1 \leqslant i, j \leqslant 2m$ the orthogonality of F_j and η imply

$$b_{ij} = \langle \nabla_{F_i}F_j, \eta\rangle = -\langle F_j, \nabla_{F_i}\eta\rangle = -\langle F_j, \nabla_{F_i}\left(\frac{a}{b}\, F_{2m} + \frac{1}{b}\, Z\right)\rangle$$

$$= -F_i\left(\frac{a}{b}\right)\langle F_j, F_{2m}\rangle - F_i\left(\frac{1}{b}\right)\langle F_j, Z\rangle - \frac{a}{b}\, \langle F_j, \nabla_{F_i}F_{2m}\rangle - \frac{1}{b}\, \langle F_j, \nabla_{F_i}Z\rangle.$$

Hence,

$$\langle \nabla_{F_i} F_{2m}, F_j \rangle = -\frac{b}{a}\, b_{ij} - \frac{b}{a}\, F_i\left(\frac{a}{b}\right)\langle F_j, F_{2m}\rangle$$
$$-\frac{b}{a}\, F_i\left(\frac{1}{b}\right)\langle F_j, Z\rangle - \frac{1}{a}\,\langle F_j, \nabla_{F_i} Z\rangle.$$

Using the fact that Z is invariant obtain from (6) and (26) the following

$$\begin{array}{lcll}
\langle \nabla_{F_i} F_{2m}, F_j \rangle & = & -\frac{b}{a}\, b_{ij}, & i, j \neq m, 2m,\ |i-j| \neq m; \\
\langle \nabla_{F_i} F_{2m}, F_{i+m} \rangle & = & -\frac{b}{a}\, b_{i\,i+m} + \frac{1}{2a}, & 1 \leqslant i \leqslant m-1; \\
\langle \nabla_{F_{j+m}} F_{2m}, F_j \rangle & = & -\frac{b}{a}\, b_{m+j\,j} - \frac{1}{2a}, & 1 \leqslant j \leqslant m-1; \\
\langle \nabla_{F_i} F_{2m}, F_m \rangle & = & -\frac{b}{a}\, b_{im}, & i \neq m, 2m; \\
\langle \nabla_{F_i} F_{2m}, F_{2m} \rangle & = & -\frac{1}{a}\, F_i(a), & i \neq m, 2m; \\
\langle \nabla_{F_m} F_{2m}, F_j \rangle & = & -\frac{b}{a}\, b_{mj}, & j \neq m, 2m; \\
\langle \nabla_{F_{2m}} F_{2m}, F_j \rangle & = & 0, & j \neq m, 2m; \\
\langle \nabla_{F_m} F_{2m}, F_m \rangle & = & -\frac{b}{a}\, b_{mm}; & \\
\langle \nabla_{F_m} F_{2m}, F_{2m} \rangle & = & \frac{b^3}{2a} - \frac{1}{a}\, F_m(a); & \\
\langle \nabla_{F_{2m}} F_{2m}, F_m \rangle & = & -\frac{b(1+a^2)}{2a}; & \\
\langle \nabla_{F_{2m}} F_{2m}, F_{2m} \rangle & = & \frac{b}{a}\, mH - \frac{1}{a}\, F_{2m}(a). &
\end{array} \tag{27}$$

It follows from $|F_{2m}| = 1$ that $\langle \nabla_{F_i} F_{2m}, F_{2m} \rangle = 0$ for $1 \leqslant i \leqslant 2m$, thus,

$$F_i(a) = 0,\ i \neq m, 2m;\ F_m(a) = \frac{b^3}{2};\ F_{2m}(a) = bmH. \tag{28}$$

The equality $a^2 + b^2 = 1$ imply $aF_i(a) + bF_i(b) = 0$ for $1 \leqslant i \leqslant 2m$. Hence,

$$\begin{gathered}
F_i(b) = 0,\ i \neq m, 2m;\ F_m(b) = -\frac{ab^2}{2};\ F_{2m}(b) = -amH; \\
F_{2m}\left(\frac{b}{a}\right) = \frac{aF_{2m}(b) - bF_{2m}(a)}{a^2} = -\frac{mH}{a^2}.
\end{gathered} \tag{29}$$

Obtain some Codazzi equations for M. For $i \neq m, 2m$

$$(\nabla_{F_{2m}} B)(F_i, F_{2m}) - (\nabla_{F_i} B)(F_{2m}, F_{2m}) = \langle R(F_{2m}, F_i)F_{2m}, \eta\rangle$$

$$= \langle R(X_{2m} - aZ, X_i)(X_{2m} - aZ), X_{2m+1} + bZ\rangle = \frac{3}{4}\ \langle J([X_{2m}, X_i])X_{2m}, X_{2m+1}\rangle$$

$$-\frac{1}{4}\ ab\langle [X_i, J(Z)X_{2m}], Z\rangle - \frac{1}{4}\ ab\langle [X_{2m}, J(-Z)X_i], Z\rangle$$

$$+\frac{1}{4}\ ab\langle [X_i, J(-Z)X_{2m}], Z\rangle + \frac{1}{4}\ a^2\langle J(Z)^2 X_i, X_{2m+1}\rangle = 0.$$

Here (7) was used. It follows that

$$0 = F_{2m}(B(F_i, F_{2m})) - B\left(\overline{\nabla}_{F_{2m}} F_i, F_{2m}\right) - B\left(F_i, \overline{\nabla}_{F_{2m}} F_{2m}\right)$$

$$-F_i(B(F_{2m}, F_{2m})) + 2B\left(\overline{\nabla}_{F_i} F_{2m}, F_{2m}\right) = F_{2m}(b_{i\,2m}) - \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{j\,2m}$$

$$-\langle \nabla_{F_{2m}} F_i, F_m\rangle b_{m\,2m} - \langle \nabla_{F_{2m}} F_i, F_{2m}\rangle b_{2m\,2m} - \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_{2m}, F_j\rangle b_{j\,i}$$

$$-\langle \nabla_{F_{2m}} F_{2m}, F_m\rangle b_{m\,i} - \langle \nabla_{F_{2m}} F_{2m}, F_{2m}\rangle b_{2m\,i} - F_i(b_{2m\,2m})$$

$$+2\sum_{j\neq m,2m} \langle \nabla_{F_i} F_{2m}, F_j\rangle b_{j\,2m} + 2\langle \nabla_{F_i} F_{2m}, F_m\rangle b_{m\,2m} + 2\langle \nabla_{F_i} F_{2m}, F_{2m}\rangle b_{2m\,2m},$$

where $\overline{\nabla}$ is the induced connection on M. Using (26) and (27) obtain

$$0 = -\frac{a^2}{2}\ \langle \nabla_{F_{2m}} F_i, F_m\rangle + \frac{b(1+a^2)}{2a}\ b_{m\,i} - abb_{m\,i} = -\frac{a^2}{2}\ \langle \nabla_{F_{2m}} F_i, F_m\rangle + \frac{b^3}{2a}\ b_{m\,i}.$$

Here, for example, $\langle \nabla_{F_{2m}} F_i, F_{2m}\rangle = -\langle \nabla_{F_{2m}} F_{2m}, F_i\rangle$ because $\langle F_{2m}, F_i\rangle = 0$. So, finally we have

$$\langle \nabla_{F_{2m}} F_i, F_m\rangle = \frac{b^3}{a^3}\ b_{m\,i}. \tag{30}$$

In the same way obtain the following Codazzi equation:

$$(\nabla_{F_{2m}} B)(F_m, F_{2m}) - (\nabla_{F_m} B)(F_{2m}, F_{2m}) = \langle R(F_{2m}, F_m)F_{2m}, \eta\rangle$$

$$= \langle R(X_{2m} - aZ, X_m)(X_{2m} - aZ), X_{2m+1} + bZ\rangle = \frac{3}{4}\ \langle J([X_{2m}, X_m])X_{2m}, X_{2m+1}\rangle$$

$$-\frac{1}{4}\ ab\langle [X_m, J(Z)X_{2m}], Z\rangle - \frac{1}{4}\ ab\langle [X_{2m}, J(-Z)X_m], Z\rangle$$

$$+\frac{1}{4}\ ab\langle [X_m, J(-Z)X_{2m}], Z\rangle + \frac{1}{4}\ a^2\langle J(Z)^2 X_m, X_{2m+1}\rangle = 0.$$

Thus,

$$0 = F_{2m}(B(F_m, F_{2m})) - B\left(\overline{\nabla}_{F_{2m}} F_m, F_{2m}\right) - B\left(F_m, \overline{\nabla}_{F_{2m}} F_{2m}\right)$$

$$-F_m(B(F_{2m}, F_{2m})) + 2B\left(\overline{\nabla}_{F_m} F_{2m}, F_{2m}\right) = F_{2m}(b_{m\,2m}) - \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_m, F_j\rangle b_{j\,2m}$$

$$-\langle \nabla_{F_{2m}} F_m, F_m\rangle b_{m\,2m} - \langle \nabla_{F_{2m}} F_m, F_{2m}\rangle b_{2m\,2m} - \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_{2m}, F_j\rangle b_{j\,m}$$

$$-\langle \nabla_{F_{2m}} F_{2m}, F_m\rangle b_{m\,m} - \langle \nabla_{F_{2m}} F_{2m}, F_{2m}\rangle b_{2m\,m} - F_m(b_{2m\,2m})$$

$$+2 \sum_{j\neq m,2m} \langle \nabla_{F_m} F_{2m}, F_j\rangle b_{j\,2m} + 2\langle \nabla_{F_m} F_{2m}, F_m\rangle b_{m\,2m} + 2\langle \nabla_{F_m} F_{2m}, F_{2m}\rangle b_{2m\,2m}.$$

It follows from (26)–(28) that

$$0 = aF_{2m}(a) + \frac{b(1+a^2)mH}{2a} + \frac{b(1+a^2)}{2a}\, b_{mm} - abb_{mm} = \frac{b(1+3a^2)mH}{2a} + \frac{b^3}{2a}\, b_{mm}.$$

Here $\langle \nabla_{F_{2m}} F_m, F_m\rangle = 0$ because $\langle F_m, F_m\rangle = 1$. Simplify and obtain

$$b_{mm} = -\frac{(1+3a^2)mH}{b^2}. \tag{31}$$

Using (28) and (29) we have

$$F_{2m}(b_{mm}) = -\frac{\left(6aF_{2m}(a)b - 2F_{2m}(b)(1+3a^2)\right)mH}{b^3} = -\frac{8am^2H^2}{b^3}. \tag{32}$$

For $i \neq m, 2m$ similarly compute

$$(\nabla_{F_i} B)(F_{2m}, F_i) - (\nabla_{F_{2m}} B)(F_i, F_i) = \langle R(F_i, F_{2m})F_i, \eta\rangle$$

$$= \langle R(X_i, X_{2m} - aZ)X_i, X_{2m+1} + bZ\rangle = \frac{3}{4}\,\langle J([X_i, X_{2m}])X_i, X_{2m+1}\rangle - \frac{1}{4}\, ab\langle [X_i, J(-Z)X_i], Z\rangle = \frac{ab}{4}.$$

Hence,

$$\frac{ab}{4} = F_i(B(F_{2m}, F_i)) - B\left(\overline{\nabla}_{F_i} F_{2m}, F_i\right) - B\left(F_{2m}, \overline{\nabla}_{F_i} F_i\right)$$

$$-F_{2m}(B(F_i, F_i)) + 2B\left(\overline{\nabla}_{F_{2m}} F_i, F_i\right) = F_i(b_{2m\,i}) - \sum_{j\neq m,2m} \langle \nabla_{F_i} F_{2m}, F_j\rangle b_{ji}$$

$$-\langle \nabla_{F_i} F_{2m}, F_m\rangle b_{mi} - \langle \nabla_{F_i} F_{2m}, F_{2m}\rangle b_{2m\,i} - \sum_{j\neq m,2m} \langle \nabla_{F_i} F_i, F_j\rangle b_{j\,2m}$$

$$-\langle \nabla_{F_i} F_i, F_m\rangle b_{m\,2m} - \langle \nabla_{F_i} F_i, F_{2m}\rangle b_{2m\,2m} - F_{2m}(b_{ii})$$

$$+2 \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ji} + 2\langle \nabla_{F_{2m}} F_i, F_m\rangle b_{mi} + 2\langle \nabla_{F_{2m}} F_i, F_{2m}\rangle b_{2m\,i}.$$

Using (26), (27), and (30) obtain

$$\frac{ab}{4} = \frac{b}{a} \sum_{j\neq m,2m} b_{ij}^2 + \frac{\varepsilon(i)}{2a}\, b_{i\,[m+i]} + \frac{b}{a}\, b_{im}^2 - \frac{a^2}{2}\,\langle \nabla_{F_i} F_i, F_m\rangle + \frac{bmH}{a}\, b_{ii}$$

$$-F_{2m}(b_{ii}) + 2\sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} + 2\frac{b^3}{a^3}\, b_{mi}^2.$$

Here $\varepsilon(i) = -1$ if $i < m$, $\varepsilon(i) = 1$ if $i > m$, and $[m+i]$ is the remainder after dividing $m+i$ by $2m$. So,

$$F_{2m}(b_{ii}) = -\frac{ab}{4} + \frac{b}{a} \sum_{j\neq m,2m} b_{ij}^2 + \frac{\varepsilon(i)}{2a}\, b_{i\,[m+i]} + \frac{b}{a}\left(1 + 2\frac{b^2}{a^2}\right) b_{im}^2$$
$$-\frac{a^2}{2}\,\langle\nabla_{F_i} F_i, F_m\rangle + \frac{bmH}{a}\, b_{ii} + 2\sum_{j\neq m,2m}\langle\nabla_{F_{2m}} F_i, F_j\rangle b_{ij}. \tag{33}$$

One more Codazzi equation is

$$(\nabla_{F_m} B)(F_{2m}, F_m) - (\nabla_{F_{2m}} B)(F_m, F_m) = \langle R(F_m, F_{2m})F_m, \eta\rangle$$

$$= \langle R(X_m, X_{2m} - aZ)X_m, X_{2m+1} + bZ\rangle = \frac{3}{4}\,\langle J([X_m, X_{2m}])X_m, X_{2m+1}\rangle$$

$$-\frac{1}{4}\, ab\langle [X_m, J(-Z)X_m], Z\rangle = ab.$$

It follows that

$$ab = F_m(B(F_{2m}, F_m)) - B\left(\overline{\nabla}_{F_m} F_{2m}, F_m\right) - B\left(F_{2m}, \overline{\nabla}_{F_m} F_m\right)$$

$$-F_{2m}(B(F_m, F_m)) + 2B\left(\overline{\nabla}_{F_{2m}} F_m, F_m\right) = F_m(b_{2m\,m}) - \sum_{j\neq m,2m}\langle\nabla_{F_m} F_{2m}, F_j\rangle b_{jm}$$
$$-\langle\nabla_{F_m} F_{2m}, F_m\rangle b_{mm} - \langle\nabla_{F_m} F_{2m}, F_{2m}\rangle b_{2m\,m} - \sum_{j\neq m,2m}\langle\nabla_{F_m} F_m, F_j\rangle b_{j\,2m}$$
$$-\langle\nabla_{F_m} F_m, F_m\rangle b_{m\,2m} - \langle\nabla_{F_m} F_m, F_{2m}\rangle b_{2m\,2m} - F_{2m}(b_{mm})$$

$$+2\sum_{j\neq m,2m}\langle\nabla_{F_{2m}} F_m, F_j\rangle b_{jm} + 2\langle\nabla_{F_{2m}} F_m, F_m\rangle b_{mm} + 2\langle\nabla_{F_{2m}} F_m, F_{2m}\rangle b_{2m\,m}.$$

Using (26)–(28) and (30)–(32) we have

$$ab = aF_m(a) + \frac{b}{a}\sum_{j\neq m,2m} b_{mj}^2 + \frac{b}{a}\, b_{mm}^2 + \frac{b}{a}\, mHb_{mm} - F_{2m}(b_{mm})$$
$$-2\frac{b^3}{a^3}\sum_{j\neq m,2m} b_{mj}^2 + \frac{ab(1+a^2)}{2} = \frac{ab^3}{2} + \frac{ab(1+a^2)}{2} + \frac{b}{a}\left(1 - 2\frac{b^2}{a^2}\right)\sum_{j\neq m,2m} b_{mj}^2$$
$$+\frac{(1+3a^2)^2 m^2 H^2}{ab^3} - \frac{(1+3a^2)m^2H^2}{ab} + \frac{8am^2H^2}{b^3}.$$

Finally obtain

$$\left(1 - 2\frac{b^2}{a^2}\right)\sum_{j\neq m,2m} b_{mj}^2 + \frac{12a^2(1+a^2)m^2H^2}{b^4} = 0. \tag{34}$$

Also, for $i \neq m,\ 2m$

$$(\nabla_{F_i} B)(F_m, F_{2m}) - (\nabla_{F_m} B)(F_i, F_{2m}) = \langle R(F_i, F_m)F_{2m}, \eta\rangle$$

$$= \langle R(X_i, X_m)(X_{2m} - aZ), X_{2m} + bZ\rangle = \frac{1}{2}\,\langle J([X_i, X_m])X_{2m}, X_{2m}\rangle$$
$$-\frac{1}{4}\,\langle J([X_m, X_{2m}])X_i, X_{2m}\rangle + \frac{1}{4}\,\langle J([X_i, X_{2m}])X_m, X_{2m}\rangle$$

$$+\frac{1}{4}\, ab\langle [X_i, J(Z)X_m], Z\rangle - \frac{1}{4}\, ab\langle [X_m, J(Z)X_i], Z\rangle = 0.$$

Thus,

$$0 = F_i(B(F_m, F_{2m})) - B\left(\overline{\nabla}_{F_i}F_m, F_{2m}\right) - B\left(F_m, \overline{\nabla}_{F_i}F_{2m}\right)$$

$$-F_m(B(F_i, F_{2m})) + B\left(\overline{\nabla}_{F_m}F_i, F_{2m}\right) + B\left(F_i, \overline{\nabla}_{F_m}F_{2m}\right) = F_i(b_{m\,2m})$$

$$- \sum_{j\neq m,2m} \langle \nabla_{F_i}F_m, F_j\rangle b_{j\,2m} - \langle \nabla_{F_i}F_m, F_m\rangle b_{m\,2m} - \langle \nabla_{F_i}F_m, F_{2m}\rangle b_{2m\,2m}$$
$$- \sum_{j\neq m,2m} \langle \nabla_{F_i}F_{2m}, F_j\rangle b_{j\,m} - \langle \nabla_{F_i}F_{2m}, F_m\rangle b_{m\,m} - \langle \nabla_{F_i}F_{2m}, F_{2m}\rangle b_{2m\,m}$$
$$-F_m(b_{i\,2m}) + \sum_{j\neq m,2m} \langle \nabla_{F_m}F_i, F_j\rangle b_{j\,2m} + \langle \nabla_{F_m}F_i, F_m\rangle b_{m\,2m}$$
$$+\langle \nabla_{F_m}F_i, F_{2m}\rangle b_{2m\,2m} + \sum_{j\neq m,2m} \langle \nabla_{F_m}F_{2m}, F_j\rangle b_{j\,i}$$
$$+\langle \nabla_{F_m}F_{2m}, F_m\rangle b_{m\,i} + \langle \nabla_{F_m}F_{2m}, F_{2m}\rangle b_{2m\,i}.$$

Equations in (26)–(28) and (31) imply

$$0 = aF_i(a) + \frac{b}{a}\, mHb_{i\,m} - \sum_{j\neq m,2m} \langle \nabla_{F_i}F_{2m}, F_j\rangle b_{j\,m} - \frac{(1+3a^2)mH}{ab}\, b_{i\,m}$$
$$+\frac{1}{2}\, a^2\langle \nabla_{F_m}F_i, F_m\rangle - \frac{b}{a}\, mHb_{i\,m} - \frac{b}{a} \sum_{j\neq m,2m} b_{j\,m}b_{j\,i} + \frac{(1+3a^2)mH}{ab}\, b_{i\,m}$$
$$= \frac{1}{2a}\, \varepsilon(i)b_{[m+i]\,m} + \frac{1}{2}\, a^2\langle \nabla_{F_m}F_i, F_m\rangle.$$

Finally we have

$$\langle \nabla_{F_m}F_i, F_m\rangle = -\frac{1}{a^3}\, \varepsilon(i)b_{[m+i]\,m}. \tag{35}$$

Take $1 \leqslant i \leqslant 2m$. Obtain the intrinsic sectional curvature $K_{i\,2m}$ of M in the direction of the plane spanned by F_i and F_{2m}. It equals

$$\langle \overline{\nabla}_{F_i}\overline{\nabla}_{F_{2m}}F_{2m}, F_i\rangle - \langle \overline{\nabla}_{F_{2m}}\overline{\nabla}_{F_i}F_{2m}, F_i\rangle - \langle \overline{\nabla}_{[F_i,F_{2m}]}F_{2m}, F_i\rangle$$

$$= F_i\left(\langle \overline{\nabla}_{F_{2m}}F_{2m}, F_i\rangle\right) - \langle \overline{\nabla}_{F_{2m}}F_{2m}, \overline{\nabla}_{F_i}F_i\rangle - F_{2m}\left(\langle \overline{\nabla}_{F_i}F_{2m}, F_i\rangle\right)$$

$$+\langle \overline{\nabla}_{F_i}F_{2m}, \overline{\nabla}_{F_{2m}}F_i\rangle - \langle \overline{\nabla}_{\overline{\nabla}_{F_i}F_{2m}}F_{2m}, F_i\rangle + \langle \overline{\nabla}_{\overline{\nabla}_{F_{2m}}F_i}F_{2m}, F_i\rangle.$$

It follows that

$$\begin{aligned}
K_{i\,2m} &= F_i\left(\langle \nabla_{F_{2m}}F_{2m}, F_i\rangle\right) - \sum_{j\neq m,2m} \langle \nabla_{F_{2m}}F_{2m}, F_j\rangle\langle \nabla_{F_i}F_i, F_j\rangle \\
&-\langle \nabla_{F_{2m}}F_{2m}, F_m\rangle\langle \nabla_{F_i}F_i, F_m\rangle - \langle \nabla_{F_{2m}}F_{2m}, F_{2m}\rangle\langle \nabla_{F_i}F_i, F_{2m}\rangle \\
&-F_{2m}\left(\langle \nabla_{F_i}F_{2m}, F_i\rangle\right) + \sum_{j\neq m,2m} \langle \nabla_{F_i}F_{2m}, F_j\rangle\langle \nabla_{F_{2m}}F_i, F_j\rangle \\
&+\langle \nabla_{F_i}F_{2m}, F_m\rangle\langle \nabla_{F_{2m}}F_i, F_m\rangle + \langle \nabla_{F_i}F_{2m}, F_{2m}\rangle\langle \nabla_{F_{2m}}F_i, F_{2m}\rangle \\
&- \sum_{j\neq m,2m} \langle \nabla_{F_i}F_{2m}, F_j\rangle\langle \nabla_{F_j}F_{2m}, F_i\rangle - \langle \nabla_{F_i}F_{2m}, F_m\rangle\langle \nabla_{F_m}F_{2m}, F_i\rangle \\
&-\langle \nabla_{F_i}F_{2m}, F_{2m}\rangle\langle \nabla_{F_{2m}}F_{2m}, F_i\rangle + \sum_{j\neq m,2m} \langle \nabla_{F_{2m}}F_i, F_j\rangle\langle \nabla_{F_j}F_{2m}, F_i\rangle \\
&+\langle \nabla_{F_{2m}}F_i, F_m\rangle\langle \nabla_{F_m}F_{2m}, F_i\rangle + \langle \nabla_{F_{2m}}F_i, F_{2m}\rangle\langle \nabla_{F_{2m}}F_{2m}, F_i\rangle.
\end{aligned} \tag{36}$$

Consider the Gauss equations for M. Let $i \neq m, 2m$. It follows from (27)–(30) that (36) takes the form

$$K_{i\,2m} = \frac{b(1+a^2)}{2a}\ \langle \nabla_{F_i} F_i, F_m\rangle + F_{2m}\left(\frac{b}{a}\ b_{ii}\right) - 2\frac{b}{a} \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij}$$
$$-\frac{b^4}{a^4}\ b_{im}^2 - \frac{b^2}{a^2} \sum_{j\neq m,2m,\,|i-j|\neq m} b_{ij}^2 - \left(-\frac{b}{a}\ b_{i\,[i+m]} - \frac{\varepsilon(i)}{2a}\right)\left(-\frac{b}{a}\ b_{i\,[i+m]} + \frac{\varepsilon(i)}{2a}\right)$$
$$-\frac{b^2}{a^2}\ b_{im}^2 - \frac{b^4}{a^4}\ b_{im}^2 = \frac{b(1+a^2)}{2a}\ \langle \nabla_{F_i} F_i, F_m\rangle + \frac{b}{a}\ F_{2m}(b_{ii}) - \frac{mH}{a^2}\ b_{ii}$$
$$-\frac{2b}{a} \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} - \frac{b^2}{a^2}\ \left(1 + 2\frac{b^2}{a^2}\right) b_{im}^2 - \frac{b^2}{a^2} \sum_{j\neq m,2m} b_{ij}^2 + \frac{1}{4a^2}.$$

It follows from (26) that the extrinsic curvature in the direction of the plane spanned by F_i and F_{2m} equals $-mHb_{ii}$. The curvature of the Heisenberg group in this direction is

$$\langle R(F_i, F_{2m})F_{2m}, F_i\rangle = \langle R(X_i, X_{2m} - aZ)(X_{2m} - aZ), X_i\rangle$$
$$= \frac{3}{4}\ \langle J([X_i, X_{2m}])X_{2m}, X_i\rangle - a^2\frac{1}{4}\ \langle J(Z)^2 X_i, X_i\rangle = \frac{a^2}{4}.$$

Hence, the Gauss equation has the form

$$-mHb_{ii} + \frac{a^2}{4} = \frac{b(1+a^2)}{2a}\ \langle \nabla_{F_i} F_i, F_m\rangle + \frac{b}{a}\ F_{2m}(b_{ii}) - \frac{mH}{a^2}\ b_{ii}$$
$$-\frac{2b}{a} \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} - \frac{b^2}{a^2}\ \left(1 + 2\frac{b^2}{a^2}\right) b_{im}^2 - \frac{b^2}{a^2} \sum_{j\neq m,2m} b_{ij}^2 + \frac{1}{4a^2}.$$

Simplifying we get

$$F_{2m}(b_{ii}) = \frac{bmH}{a}\ b_{ii} - \frac{b(1+a^2)}{4a} - \frac{(1+a^2)}{2}\ \langle \nabla_{F_i} F_i, F_m\rangle$$
$$+2 \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} + \frac{b}{a}\ \left(1 + 2\frac{b^2}{a^2}\right) b_{im}^2 + \frac{b}{a} \sum_{j\neq m,2m} b_{ij}^2. \tag{37}$$

Compare (33) with (37) and obtain

$$-\frac{b}{4a} - \frac{1}{2}\ \langle \nabla_{F_i} F_i, F_m\rangle = \frac{\varepsilon(i)}{2a}\ b_{i\,[m+i]}. \tag{38}$$

It follows from (37) and (38) that

$$F_{2m}(b_{ii}) = \frac{bmH}{a}\ b_{ii} + \frac{\varepsilon(i)(1+a^2)}{2a}\ b_{i\,[m+i]}$$
$$+2 \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} + \frac{b}{a}\ \left(1 + 2\frac{b^2}{a^2}\right) b_{im}^2 + \frac{b}{a} \sum_{j\neq m,2m} b_{ij}^2. \tag{39}$$

Definition of H, (26), and (31) imply

$$2mH = \sum_{i\neq m,2m} b_{ii} - \frac{(1+3a^2)mH}{b^2} - mH. \tag{40}$$

H is constant, so (32), (39), and (40) imply

$$
\begin{aligned}
0 = F_{2m}(2mH) = \sum_{i\neq m,2m} F_{2m}(b_{ii}) - \frac{8am^2H^2}{b^3} = \frac{bmH}{a} \sum_{i\neq m,2m} b_{ii} \\
+ \sum_{i\neq m,2m} \left(\frac{\varepsilon(i)(1+a^2)}{2a}\, b_{i\,[m+i]} + 2 \sum_{j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} \right. \\
\left. + \frac{b}{a}\left(1 + 2\frac{b^2}{a^2}\right) b_{im}^2 + \frac{b}{a} \sum_{j\neq m,2m} b_{ij}^2 \right) - \frac{8am^2H^2}{b^3}.
\end{aligned}
\tag{41}
$$

Note that

$$\sum_{i\neq m,2m} \frac{\varepsilon(i)(1+a^2)}{2a}\, b_{i\,[m+i]} = \sum_{i,j\neq m,2m} \langle \nabla_{F_{2m}} F_i, F_j\rangle b_{ij} = 0.$$

It follows from (40) and (41) that

$$0 = \frac{bmH}{a}\left(3mH + \frac{(1+3a^2)mH}{b^2}\right) + \frac{b}{a}\left(1 + 2\frac{b^2}{a^2}\right) \sum_{i\neq m,2m} b_{im}^2 + \frac{b}{a} \sum_{i,j\neq m,2m} b_{ij}^2 - \frac{8am^2H^2}{b^3}.$$

Simplify and obtain

$$\left(1 + 2\frac{b^2}{a^2}\right) \sum_{i\neq m,2m} b_{im}^2 + \sum_{i,j\neq m,2m} b_{ij}^2 + \frac{4(1-3a^2)m^2H^2}{b^4} = 0. \tag{42}$$

Summing up (34) and (42) we have

$$2 \sum_{i\neq m,2m} b_{im}^2 + \sum_{i,j\neq m,2m} b_{ij}^2 + \frac{4(1+3a^4)m^2H^2}{b^4} = 0. \tag{43}$$

It follows from (26), (31), and (43) that $H = 0$ and all the coefficients of B vanish except of $b_{m\,2m} = \frac{a^2}{2}$. Then (27) and (35) imply that $\langle \nabla_{F_m} F_m, F_i\rangle = -\langle \nabla_{F_m} F_i, F_m\rangle = 0$ for all $1 \leqslant i \leqslant 2m$. Also, $\langle \nabla_{F_m} F_m, \eta\rangle = b_{mm} = 0$, hence $\nabla_{F_m} F_m = 0$ and the field F_m is geodesic in the Heisenberg group. Consider the set of geodesics in this group (see [6, Proposition (3.1), Proposition (3.5)]) and obtain that $F_m = T_m = X_m$ and $T_{2m} = X_{2m}$ are left invariant. It follows from (28) and (29) that $F_{2m}(a) = F_{2m}(b) = 0$. Thus,

$$\nabla_{F_{2m}} F_{2m} = \nabla_{F_{2m}} (bX_{2m} - aZ) = b\nabla_{bX_{2m}-aZ} X_{2m} - a\nabla_{bX_{2m}-aZ} Z = -abX_m.$$

Then it follows from (27) that $-b(1+a^2)/2a = -ab$ and $a^2 = 1$, $b = 0$, a contradiction.

So, $ab = 0$ at each point of M. Since $a^2+b^2 = 1$ and a, b are smooth, $a = 1$ or $b = 1$ identically on each connected component of M. The latter case is impossible because the distribution $Z^\perp$ is not integrable. Then $F_{2m} = \pm Z$, and M is cylindrical. □

This theorem was previously proved for the case $m = 1$ in [5]. A similar result for another definition of the Gauss map was proved in [7].

Theorem 8 and Corollary 7 imply that a complete constant mean curvature hypersurface with the harmonic Gauss map is the direct product of constant mean curvature hypersurface in E^{2m} and a straight vertical line. In particular, complete constant mean curvature surfaces with the harmonic Gauss map in the three-dimensional Heisenberg group are vertical Euclidean planes and vertical Euclidean cylinders over circles.

Acknowledgement

The author would thank prof. A.A. Borisenko for his constant attention and encouragement during the work on this paper.

References

[1] E.A. Ruh, J. Vilms, The tension field of the Gauss map, Trans. Amer. Math. Soc. 149(1970), 569-573.
[2] J.J. Eells, H. Sampson, Harmonic mappings of Riemannian manifolds, American Journal of Math. 86(1964), 109-160.
[3] N. do Espirito-Santo, S. Fornari, K. Frensel, J. Ripoll, Constant mean curvature hypersurfaces in a Lie group with a bi-invariant metric, Manuscripta Math. 111(2003), 459-470.
[4] Ye.V. Petrov, Submanifolds with the harmonic Gauss Map in Lie groups, Journ. of Math. Phys., An., Geom. 4(2008), 278-293.
[5] Ye.V. Petrov, The Gauss map of hypersurfaces in 2-step nilpotent Lie groups, Journ. of Math. Phys., An., Geom. 2(2006), 186-206.
[6] P.B. Eberlein, Geometry of 2-step nilpotent groups with a left-invariant metric, Ann. Sci. École Norm. Sup. 27(1994), 611-660.
[7] A. Sanini, Gauss map of a surface of the Heisenberg group, Boll. Unione Mat. It. 11-B(7)(1997), suppl. fasc. 2, 79-93.