Powers of the space forms curvature operator and geodesics of the tangent bundle.*

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Abstract

It is well-known that if Γ is a geodesic line of the tangent (sphere) bundle with Sasaki metric of a locally symmetric Riemannian manifold then the projected curve $\gamma = \pi \circ \Gamma$ has all its geodesic curvatures constant. In this paper we consider the case of tangent (sphere) bundle over the real, complex and quaternionic space form and give a unified proof of the following property: all geodesic curvatures of projected curve are zero starting from k_3, k_6 and k_{10} for the real, complex and quaternionic space formes respectively.

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Introduction

Sato K. [4] and Sasaki S. [3] proved that the projection to the base space of any non-vertical geodesic line on the tangent or the tangent sphere bundle of a real space form $M^n(c)$ is a curve of constant curvatures k_1 and k_2 and zero curvatures $k_3, \dots, k_{n-1}$. Nagy P. [2] essentially generalized this result. He considered the case of general locally symmetric base manifold and have proved that the geodesic curvatures of projection of any (non-vertical) geodesic line on the tangent sphere bundle are all constant. Nevertheless it was still interesting to find a clearer description of projections of geodesics for the case of classical rank one symmetric spaces. The second author made a first step in this defection and proved that the projection to the base space of any non-vertical geodesic line on the tangent or tangent sphere bundle of a complex space form CP^n is a curve of constant curvatures $k_1, \dots, k_5$ and zero curvatures $k_6, \dots, k_{n-1}$

In is this paper we make a contribution in more clear understanding of geometry of projected geodesics in the case of tangent (sphere) bundle of almost all classical locally symmetric spaces, namely *spheres, complex and quaternionic projective spaces and their non-compact dual* from a unified

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viewpoint using the *recurrent properties* of powers of the curvature operator of these spaces. This approach allows to give also a unified proof of the results from [3], [4] and [5]

We also use an easy to prove result [1], stating that the geodesics of tangent or tangent sphere bundle with Sasaki metric have the same projections to the base manifold.

Remark on notations. Throughout the paper $\langle\cdot,\cdot\rangle$ and $|\cdot|$ mean the scalar product and the norm of vectors with respect to the corresponding metrics.

1 Summary of main results.

Let $(M^n(c), g)$ be a Riemannian manifold of constant curvature c, $(M^{2n}(c); J; g)$ a Riemannian manifold with complex structure J of constant holomorphic curvature c and $(M^{4n}(c); J_1, J_2, J_3; g)$ a Riemannian manifold with quaternionic structure (J_1, J_2, J_3) of constant quaternionic curvature c. For the sake of brevity, denote by $\mathcal{M}(c)$ one of these space forms with corresponding standard metrics and will refer to $\mathcal{M}(c)$ just to a space form of constant curvature c. The main result is the following statement.

Theorem 1.1 *Let $\mathcal{M}(c)$ be a space form of constant curvature $c \neq 0$. Let Γ be non-vertical geodesic line on the tangent or tangent sphere bundle over $\mathcal{M}(c)$. Let $\gamma = \pi \circ \Gamma$ be the projection of Γ to $\mathcal{M}(c)$. Then the geodesic curvatures $k_1, k_2, \dots$ of γ are all constant and*

(a) $k_3 = \ldots = k_{n-1} = 0$ *for the real space form;*
(b) $k_6 = \ldots = k_{2n-1} = 0$ *for the complex space form;*
(c) $k_{10} = \ldots = k_{4n-1} = 0$ *for the quaternionic space form.*

As the referee remarked, the result of the Theorem 1.1 can be expressed in more clear geometrical terms, namely *the projected curve $\gamma = \pi \circ \Gamma$ lies in a totally geodesic S^3 or H^3, in a totally geodesic CP^3 or CH^3 and in a totally geodesic QP^3 or QH^3 for the real, complex and quaternionic space form respectively.* These assertions can be derived from (6), (10) and (14).

Proof of the Theorem 1.1 is based on the recurrent property of powers of curvature operator of spaces under consideration. Let R_{XY} be the curvature operator of $\mathcal{M}(c)$. Define a power of curvature operator R^p_{XY} recurrently in the following way:

$$R^p_{XY}Z = R^{p-1}_{XY}(R_{XY}Z) \quad p > 1.$$

The basic tool for our considerations are a chain of lemmas.

Lemma 1.1 *Let R_{XY} be the curvature operator of the real space form $(M^n(c), g)$. Then for any X and Y*

$$R^p_{XY} = \begin{cases} (-b^2c^2)^{s-1} R_{XY} \text{ for } p=2s-1 \\ (-b^2c^2)^{s-1} R^2_{XY} \text{ for } p=2s, \end{cases} \quad s \geq 1$$

where $b = |X \wedge Y|$ is a norm of bivector $X \wedge Y$.

Lemma 1.2 *Let R_{XY} be the curvature operator of the non-flat complex space form $(M^n(c); J; g)$. Denote by $b = |X \wedge Y|$ the norm of a bivector $X \wedge Y$ and $m = \langle X, JY \rangle$. Then for any X and Y*

$$R^p_{XY} = \begin{cases} \mathrm{Lin}\,(JR^2_{XY},\, R_{XY},\, J\,) \text{ for } p=2s-1 \\ \mathrm{Lin}\,(R^2_{XY},\, JR_{XY},\, E\,) \text{ for } p=2s, \end{cases} \quad s \geq 2$$

where E is the identity operator and Lin *means a linear combination of corresponding operators with coefficients being polynomials in $\frac{1}{c}\,, b\,, m$.*

Lemma 1.3 *Let R_{XY} be the curvature operator of the non-flat quaternionic space form $(M^n(c); J_1, J_2, J_3; g)$. Denote by $b = |X \wedge Y|$ the norm of a bivector $X \wedge Y$. Set $m_1 = \langle X, J_1Y \rangle$, $m_2 = \langle X, J_2Y \rangle$, $m_3 = \langle X, J_3Y \rangle$, $m^2 = m_1^2 + m_2^2 + m_3^2$, $\mathcal{J} = m_1J_1 + m_2J_2 + m_3J_3$. Then for any X and Y*

$$R^p_{XY} = \begin{cases} \mathrm{Lin}\,(\mathcal{J}R^4_{XY},\, R^3_{XY},\, \mathcal{J}R^2_{XY},\, R_{XY}, \mathcal{J}\,) \text{ for } p=2s-1 \\ \mathrm{Lin}\,(R^4_{XY},\, \mathcal{J}R^3_{XY},\, R^2_{XY},\, \mathcal{J}R_{XY}, E\,) \text{ for } p=2s, \end{cases} \quad s \geq 3$$

where E is the identity operator and Lin *means a linear combination of corresponding operators with coefficients being polynomials in $\frac{1}{c}\,, b\,, m$.*

2 Necessary facts and proof of the main result.

Let (M^n, g) be a Riemannian manifold and TM^n be its tangent bundle. Denote by $(u^1, \ldots, u^n)$ a local coordinate system on M^n. Then in each tangent space of M^n the natural coordinate frame $\{\partial/\partial u^1, \ldots, \partial/\partial u^n\}$ form a local basis. Let ξ be any tangent vector over the given local chart. Then ξ can be decomposed as

$$\xi = \xi^1 \partial/\partial u^1 + \ldots + \xi^n \partial/\partial u^n.$$

The parameters $(u^1, \ldots, u^n; \xi^1, \ldots, \xi^n)$ form the so called *natural induced coordinate system* in TM^n. The *Sasaki metric* line element $d\sigma^2$ with respect to this coordinate system is

$$d\sigma^2 = ds^2 + |D\xi|^2, \tag{1}$$

where ds^2 is a line element of M^n, $D\xi$ is the covariant differential of ξ with respect to Levi-Civita connection on M^n and $|\cdot|$ means the norm with respect to Riemannian metric on M^n.

The *tangent sphere bundle* T_1M^n can be considered as a hypersurface in the tangent bundle defined by the condition $|\,\xi\,| = 1$. We will consider T_1M^n as a submanifold in TM^n with the induced metric.

With respect to the natural coordinate system, each curve Γ on TM^n can be represented as $\Gamma(\sigma) = \Big\{u^1(\sigma)\dots, u^n(\sigma); \xi^1(\sigma), \dots, \xi^n(\sigma)\Big\}$ with respect to the arc-length parameter σ and can be interpreted as the vector field $\xi(\sigma) = \xi^1(\sigma)\partial/\partial u^1 + \dots + \xi^n(\sigma)\partial/\partial u^n$ along the *projected* curve $\gamma = \pi\circ\Gamma = (u^1(\sigma), \dots, u^n(\sigma))$. If ξ is a *unit* vector field then Γ lies in T_1M^n and represents an arbitrary curve in T_1M^n.

Denote by $(')$ the covariant derivative along γ with respect to parameter σ. Then Γ is a geodesic line on TM^n or T_1M^n if γ and ξ satisfy respectively the system of equations

$$TM^n : \begin{cases} \gamma'' = R_{\xi'\,\xi}\,\gamma', \\ \xi'' = 0; \end{cases} \qquad T_1M^n : \begin{cases} \gamma'' = R_{\xi'\,\xi}\,\gamma', \\ \xi'' = -\rho^2\xi, \end{cases} \tag{2}$$

where $\rho^2 = |\,\xi'\,|^2$ and $R_{\xi'\,\xi}$ is the *curvature operator* of M^n based on bivector $\xi'\wedge\xi$.

From (2) it follows that $\rho = const$ in both cases. Denote by s the arc-length parameter on γ. Then from (1) it follows that

$$\frac{ds}{d\sigma} = \sqrt{1-\rho^2}, \tag{3}$$

so that $0 \le \rho \le 1$. According to the latter inequality, the set of geodesics of TM^n and T_1M^n can be splitted naturally into 3 classes, namely,

- *horizontal* geodesics ($\rho = 0$) generated by parallel (unit) vector fields along the geodesics on the base manifold;
- *vertical* geodesics ($\rho = 1$) represented by geodesics on a fixed fiber;
- *umbilical* geodesics corresponding to $0 < \rho < 1$.

In what follows we will consider the properties of projections of umbilical geodesics.

Lemma 2.1 *(cf. [2]) Let (M^n, g) be a locally symmetric Riemannian manifold and R_{XY} its curvature operator. Let $\gamma = \pi\circ\Gamma$ be a projection of geodesic line on TM^n or T_1M^n to the base space. Then for the derivatives of γ of order p we have*

$$\gamma^{(p)} = R^{p-1}_{\xi'\,\xi}\gamma' = R_{\xi'\,\xi}\gamma^{(p-1)}$$

and as a consequence all the geodesic curvatures of γ are constant.

Proof. The equalities follow from parallelism of curvature tensor of M^n and the equations (2). Moreover, from the evident identity

$$\langle \gamma^{(p)}, \gamma^{(p-1)} \rangle \equiv 0$$

for all $p > 1$, we conclude that $|\gamma^{(p)}| = const$ for all $p > 1$ and therefore, by induction, all the geodesic curvatures of γ are constant.

■

Proof of Theorem 1.1. Case (a). Denote by $e_1, \dots, e_{n-1}$ the Frenet frame of γ. Using the Frenet formulas for the curve with constant geodesic curvatures and keeping in mind (3), it is easy to see that

$$\begin{aligned} \gamma^{(2s-1)} &= (1-\rho^2)^{s-1/2} k_1 k_2 \dots k_{2s-2}\, e_{2s-1} + \mathrm{Lin}\,\{e_1, e_3, \dots, e_{2s-3}\}, \\ \gamma^{(2s)} &= (1-\rho^2)^{s} k_1 k_2 \dots k_{2s-1}\, e_{2s} + \mathrm{Lin}\,\{e_2, e_4, \dots, e_{2s-2}\} \end{aligned} \tag{4}$$

for all $s \geq 1$ (with formal setting $k_0 \equiv 1$). Setting $s = 1, 2$ in even derivatives, we see that

$$\begin{aligned} \gamma^{(2)} &= (1-\rho^2) k_1\, e_2 \\ \gamma^{(4)} &= (1-\rho^2)^2 k_1 k_2 k_3\, e_4 + \mathrm{Lin}\,(e_2). \end{aligned} \tag{5}$$

On the other hand, applying Lemma 2.1, Lemma 1.1 and Lemma 2.1 again, we get

$$\gamma^{(4)} = R^3_{\xi'\,\xi} \gamma' = -b^2 c^2\, R_{\xi'\,\xi} \gamma' = -b^2 c^2 \gamma^{(2)}. \tag{6}$$

Substitution from (5) gives

$$(1-\rho^2)^2 k_1 k_2 k_3\, e_4 + \mathrm{Lin}\,(e_2) = 0$$

and therefore $k_3 = 0$, which completes the proof.

Remark, that b^2 is constant along γ since

$$(b^2)' = \Big(|\xi' \wedge \xi|^2\Big) = \Big(\rho^2\, |\, \xi\, |^2 - \langle \xi', \xi \rangle^2\Big)' = 2\rho^2 \langle \xi', \xi \rangle - 2\langle \xi', \xi \rangle \rho^2 \equiv 0.$$

Case (b). Denote by $e_1, \dots, e_{2n-1}$ the Frenet frame of γ. Similar to the case (a) considerations, the Frenet formulas give

$$\begin{aligned} \gamma^{(2s-1)} &= (1-\rho^2)^{s-1/2} k_1 k_2 \dots k_{2s-2}\, e_{2s-1} + \mathrm{Lin}\,\{e_1, e_3, \dots, e_{2s-3}\}, \\ \gamma^{(2s)} &= (1-\rho^2)^{s} k_1 k_2 \dots k_{2s-1}\, e_{2s} + \mathrm{Lin}\,\{e_2, e_4, \dots, e_{2s-2}\} \end{aligned} \tag{7}$$

for all $s \geq 1$. Setting $s = 1, 2, 3, 4$ in odd derivatives, we get

$$\begin{aligned} \gamma' &= (1-\rho^2)^{1/2} e_1, \\ \gamma^{(3)} &= (1-\rho^2)^{3/2} k_1 k_2\, e_3 + \mathrm{Lin}\,(e_1), \\ \gamma^{(5)} &= (1-\rho^2)^{5/2} k_1 \dots k_4\, e_5 + \mathrm{Lin}\,(e_1, e_3), \\ \gamma^{(7)} &= (1-\rho^2)^{7/2} k_1 \dots k_6\, e_7 + \mathrm{Lin}\,(e_1, e_3, e_5). \end{aligned} \tag{8}$$

On the other hand, applying Lemma 2.1, Lemma 1.2 and Lemma 2.1 again, we get

$$\begin{cases} \gamma^{(5)} = R^4_{\xi'\,\xi}\gamma' = \mathrm{Lin}\,(R^2_{\xi'\,\xi}, JR_{\xi'\,\xi}, E)\gamma' = \mathrm{Lin}\,(\gamma^{(3)}, J\gamma^{(2)}, \gamma'), \\ \gamma^{(7)} = R^6_{\xi'\,\xi}\gamma' = \mathrm{Lin}\,(R^2_{\xi'\,\xi}, JR_{\xi'\,\xi}, E)\gamma' = \mathrm{Lin}\,(\gamma^{(3)}, J\gamma^{(2)}, \gamma'), \end{cases} \tag{9}$$

Excluding $J\gamma^{(2)}$ from (9), we come to the equation

$$\gamma^{(7)} = \mathrm{Lin}\,(\gamma^{(5)}, \gamma^{(3)}, \gamma'). \tag{10}$$

Substitution from (8) imply

$$(1-\rho^2)^{7/2}k_1 \dots k_6\, e_7 + \mathrm{Lin}\,(e_1, e_3, e_5) = 0$$

and we conclude that $k_6 = 0$ which completes the proof.

Remark, that the coefficients of all linear combinations are constants. Indeed, by Lemma 1.2 the coefficients are polynomials in $1/c, b = |\xi' \wedge \xi|$ and $m = \langle \xi', J\xi\rangle$. The value b is constant along γ by the same reasons as in case (a). The value m is constant along γ since

$$m' = \langle \xi', J\xi\rangle' = \langle \xi'', J\xi\rangle + \langle \xi', J\xi'\rangle \equiv 0.$$

Case (c). Denote by $e_1, \dots, e_{4n-1}$ the Frenet frame of γ. As above, the Frenet formulas give

$$\begin{aligned} \gamma^{(2s-1)} &= (1-\rho^2)^{s-1/2}k_1k_2\dots k_{2s-2}\, e_{2s-1} + \mathrm{Lin}\,\{e_1, e_3, \dots, e_{2s-3}\}, \\ \gamma^{(2s)} &= (1-\rho^2)^{s}k_1k_2\dots k_{2s-1}\, e_{2s} + \mathrm{Lin}\,\{e_2, e_4, \dots, e_{2s-2}\} \end{aligned} \tag{11}$$

for all $s \geq 1$. Setting $s = 1, 2, 3, 4, 5, 6$ in odd derivatives, we get

$$\begin{aligned} \gamma' &= (1-\rho^2)^{1/2}e_1, \\ \gamma^{(3)} &= (1-\rho^2)^{3/2}k_1k_2\, e_3 + \mathrm{Lin}\,(e_1), \\ \gamma^{(5)} &= (1-\rho^2)^{5/2}k_1\dots k_4\, e_5 + \mathrm{Lin}\,(e_1, e_3), \\ \gamma^{(7)} &= (1-\rho^2)^{7/2}k_1\dots k_6\, e_7 + \mathrm{Lin}\,(e_1, e_3, e_5), \\ \gamma^{(9)} &= (1-\rho^2)^{9/2}k_1\dots k_8\, e_9 + \mathrm{Lin}\,(e_1, e_3, e_5, e_7), \\ \gamma^{(11)} &= (1-\rho^2)^{11/2}k_1\dots k_{10}\, e_{11} + \mathrm{Lin}\,(e_1, e_3, e_5, e_7, e_9). \end{aligned} \tag{12}$$

Applying again Lemma 2.1, Lemma 1.3 and then Lemma 2.1, we get

$$\begin{cases} \gamma^{(7)} = R^6_{\xi'\,\xi}\gamma' = & \mathrm{Lin}\,(R^4_{\xi'\,\xi}, \mathcal{J}R^3_{\xi'\,\xi}, R^2_{\xi'\,\xi}, \mathcal{J}R_{\xi'\,\xi}, E)\gamma' = \\ & \mathrm{Lin}\,(\gamma^{(5)}, \mathcal{J}\gamma^{(4)}, \gamma^{(3)}, \mathcal{J}\gamma^{(2)}, \gamma'), \\ \gamma^{(9)} = R^8_{\xi'\,\xi}\gamma' = & \mathrm{Lin}\,(R^4_{\xi'\,\xi}, \mathcal{J}R^3_{\xi'\,\xi}, R^2_{\xi'\,\xi}, \mathcal{J}R_{\xi'\,\xi}, E)\gamma' = \\ & \mathrm{Lin}\,(\gamma^{(5)}, \mathcal{J}\gamma^{(4)}, \gamma^{(3)}, \mathcal{J}\gamma^{(2)}, \gamma'), \\ \gamma^{(11)} = R^{10}_{\xi'\,\xi}\gamma' = & \mathrm{Lin}\,(R^4_{\xi'\,\xi}, \mathcal{J}R^3_{\xi'\,\xi}, R^2_{\xi'\,\xi}, \mathcal{J}R_{\xi'\,\xi}, E)\gamma' = \\ & \mathrm{Lin}\,(\gamma^{(5)}, \mathcal{J}\gamma^{(4)}, \gamma^{(3)}, \mathcal{J}\gamma^{(2)}, \gamma'). \end{cases} \tag{13}$$

Excluding $\mathcal{J}\gamma^{(2)}$ and $\mathcal{J}\gamma^{(4)}$ from (13), we come to the equation

$$\gamma^{(11)} = \mathrm{Lin}\,(\gamma^{(9)}, \gamma^{(7)}, \gamma^{(5)}, \gamma^{(3)}, \gamma'). \tag{14}$$

Substitution from (12) imply

$$(1-\rho^2)^{11/2} k_1 \dots k_{10}\, e_{11} + \mathrm{Lin}\,(e_1, e_3, e_5, e_7, e_9) = 0$$

and we conclude that $k_{10} = 0$ which completes the proof.

Remark, that the coefficients of all linear combinations are constants. Indeed, by Lemma 1.3 the coefficients are polynomials in $1/c, b = |\xi' \wedge \xi|$ and $m = \sqrt{m_1^2 + m_2^2 + m_3^2}$. The value b is constant along γ by the same reasons as in case (a). The values m_1, m_2, m_3 are all constant along γ since

$$m_i' = \langle \xi', J_i \xi\rangle' = \langle \xi'', J_i \xi\rangle + \langle \xi', J_i \xi'\rangle \equiv 0$$

for $i = 1, 2, 3$.

3 Proofs of basic Lemmas

Proof of Lemma 1.1.The curvature operator R_{XY} of the real space form $(M^n(c), g)$ has the following expression

$$R_{XY}Z = c\Big[\langle Y, Z\rangle X - \langle X, Z\rangle Y\Big].$$

Then
$R_{XY}^2 Z = c\Big[\langle Y, R_{XY}Z\rangle X - \langle X, R_{XY}Z\rangle Y\Big] = c^2\Big[\Big\langle Y, \langle Y, Z\rangle X - \langle X, Z\rangle Y\Big\rangle X - \Big\langle X, \langle Y, Z\rangle X - \langle X, Z\rangle Y\Big\rangle Y\Big] = c^2\Big[\langle Y, Z\rangle\langle X, Y\rangle X - \langle X, Z\rangle |Y|^2 X - \langle Y, Z\rangle |X|^2 Y + \langle X, Z\rangle\langle X, Y\rangle Y\Big] = c^2\Big[\langle Y, Z\rangle\Big(\langle X, Y\rangle X - |X|^2 Y\Big) + \langle X, Z\rangle\Big(\langle X, Y\rangle Y - |Y|^2 X\Big)\Big] = c\Big[\langle Y, Z\rangle R_{XY}X + \langle X, Z\rangle R_{YX}Y\Big]$.
Therefore,
$R_{XY}^3 Z = c\Big[\langle Y, R_{XY}Z\rangle R_{XY}X + \langle X, R_{XY}Z\rangle R_{YX}Y\Big] = c^3\Big[\Big(\langle Y, Z\rangle\langle X, Y\rangle - \langle X, Z\rangle |Y|^2\Big)\Big(\langle X, Y\rangle X - |X|^2 Y\Big) + \Big(\langle Y, Z\rangle |X|^2 - \langle X, Z\rangle\langle X, Y\rangle\Big)\Big(\langle X, Y\rangle Y - |Y|^2 X\Big)\Big] = c^3\Big[-\langle Y, Z\rangle X\Big(|X|^2|Y|^2 - \langle X, Y\rangle^2\Big) + \langle X, Z\rangle Y\Big(|X|^2|Y|^2 - \langle X, Y\rangle^2\Big)\Big] = -c^2 b^2 R_{XY}Z$,
where, evidently, $b^2 = |X|^2|Y|^2 - \langle X, Y\rangle^2$ is the square norm of $X \wedge Y$.

Now we can find the other powers for R_{XY} inductively.

■

Proof of Lemma 1.2

The curvature operator R_{XY} of the complex space form $(M^{2n}(c); J; g)$ has the following expression

$$R_{XY}Z = \frac{c}{4}\Big[\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\,\langle X, JY\rangle JZ\Big].$$

Introduce the unit sphere type operator S acting as

$$S(Z) \stackrel{def}{=} S_{XY}Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y,$$

and the operator $\hat{S}(Z)$ acting as

$$\hat{S}(Z) \stackrel{def}{=} S_{JX\,JY}Z = \langle JY, Z\rangle JX - \langle JX, Z\rangle JY.$$

Finally, if we denote $m = \langle X, JY\rangle$, then the curvature operator under consideration takes the form

$$R_{XY}Z = \frac{c}{4}\Big[S + \hat{S} + 2m\,J\Big]Z. \tag{15}$$

Since $|X \wedge Y| = |(JX) \wedge (JY)|$, the operators S and $\hat{S}$ satisfy

$$S^3 = -b^2 S, \quad \hat{S}^3 = -b^2 \hat{S},$$

where $b^2 = |X \wedge Y|^2$.

In what follows, we need a "table of products" for the operators S and $\hat{S}$. Namely,

	S	$\hat{S}$	J
S	S^2	$mJ\hat{S}$	$J\hat{S}$
$\hat{S}$	mJS	$\hat{S}^2$	JS
J	JS	$J\hat{S}$	$-E$

(16)

Indeed,

$$\begin{aligned}
&(S\hat{S})(Z) = S_{XY}\hat{S}_{JX\,JY}Z = S_{XY}\Big[\langle JY, Z\rangle JX - \langle JX, Z\rangle JY\Big] = \\
&\Big\langle Y, \langle JY, Z\rangle JX - \langle JX, Z\rangle JY\Big\rangle X - \Big\langle X, \langle JY, Z\rangle JX - \langle JX, Z\rangle JY\Big\rangle Y = \\
&\langle Y, JX\rangle\langle JY, Z\rangle X + \langle JX, Z\rangle\langle X, JY\rangle Y = \\
&\qquad\qquad mJ\Big[\langle JY, Z\rangle JX - \langle JX, Z\rangle JY\Big] = (mJ\hat{S})(Z), \\
&(\hat{S}S)(Z) = S_{JX\,JY}S_{XY}Z = S_{JX\,JY}\Big[\langle Y, Z\rangle X - \langle X, Z\rangle Y\Big] = \\
&\Big\langle JY, \langle Y, Z\rangle X - \langle X, Z\rangle Y\Big\rangle JX - \Big\langle JX, \langle Y, Z\rangle X - \langle X, Z\rangle Y\Big\rangle JY = \\
&\langle JY, X\rangle\langle Y, Z\rangle JX + \langle X, Z\rangle\langle JX, Y\rangle JY = \\
&\qquad\qquad mJ\Big[\langle Y, Z\rangle X - \langle X, Z\rangle Y\Big] = (mJS)(Z), \\
&(SJ)(Z) = S_{XY}JZ = \langle Y, JZ\rangle X - \langle X, JZ\rangle Y = \\
&\qquad\qquad J[\langle JY, Z\rangle JX - \langle JX, Z\rangle JY] = (J\hat{S})(Z), \\
&(\hat{S}J)(Z) = S_{JX\,JY}JZ = \langle JY, JZ\rangle JX - \langle JX, JZ\rangle JY = \\
&\qquad\qquad J[\langle Y, Z\rangle X - \langle X, Z\rangle Y] = (JS)(Z),
\end{aligned}$$

and the other entries of the table can be found in a similar way.

From (16) we see that $J(S+\hat{S}) = (S+\hat{S})J$ and we have

$$
\begin{aligned}
(S+\hat{S})^2 &= S^2+\hat{S}^2+S\hat{S}+\hat{S}S = S^2+\hat{S}^2+mJ(S+\hat{S})\\
(S+\hat{S})^3 &= (S+\hat{S})[S^2+\hat{S}^2+mJ(S+\hat{S})] =\\
& S^3+\hat{S}^3+\hat{S}S^2+S\hat{S}^2+mJ(S+\hat{S})^2 =\\
& -b^2S-b^2\hat{S}+(\hat{S}S)S+(S\hat{S})\hat{S}+mJ(S+\hat{S})^2 =\\
& -b^2(S+\hat{S})+mJ(S^2+\hat{S}^2)+mJ(S+\hat{S})^2 =\\
& -b^2(S+\hat{S})+mJ[(S+\hat{S})^2-mJ(S+\hat{S})]+\\
& mJ(S+\hat{S})^2 = (m^2-b^2)(S+\hat{S})+2mJ(S+\hat{S})^2.
\end{aligned}
$$

Thus,

$$(S+\hat{S})^3 = \mathrm{Lin}\,(S+\hat{S}, J(S+\hat{S})^2) \tag{17}$$

On the other hand, setting for brevity $R_{XY} = R$, from (15) we derive

$$
\begin{aligned}
& S+\hat{S} = \tfrac{4}{c}R - 2mJ = \mathrm{Lin}\,(R,J),\\
& (S+\hat{S})^2 = \mathrm{Lin}\,(R^2, JR, E).
\end{aligned} \tag{18}
$$

Comparing (17) and (18) we conclude

$$(S+\hat{S})^3 = \mathrm{Lin}\,\Big[\mathrm{Lin}\,(R,J), J\,\mathrm{Lin}\,(R^2,JR,E)\Big] = \mathrm{Lin}\,(JR^2,R,J).$$

On the other hand, from $(18)_1$

$$(S+\hat{S})^3 = \left(\frac{4}{c}\right)^3 R^3 + \mathrm{Lin}\,(JR^2,R,J).$$

So, finally

$$R^3 = \mathrm{Lin}\,(JR^2,R,J).$$

It is easy to trace that the coefficients of all linear combinations are polynomials in $\frac{1}{c}, b, m$. To complete the proof we should remark that

$$
\begin{aligned}
R^4 = R^3R = \ & \mathrm{Lin}\,(JR^2,R,J)R = \mathrm{Lin}\,(JR^3,R^2,JR) =\\
& \mathrm{Lin}\,\Big[J\,\mathrm{Lin}\,(JR^2,R,J),R^2,JR\Big] = \mathrm{Lin}\,(R^2,JR,E)
\end{aligned}
$$

which allows to find all powers of R inductively.

■

Proof of Lemma 1.3

The curvature operator R_{XY} of the quaternionic space form $(M^{4n}(c); J_1, J_2, J_3; g)$ has the following expression

$$
\begin{aligned}
R_{XY}Z = \ & \tfrac{c}{4}\Big[\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle J_1Y,Z\rangle J_1X - \langle J_1X,Z\rangle J_1Y +\\
& \langle J_2Y,Z\rangle J_2X - \langle J_2X,Z\rangle J_2Y + \langle J_3Y,Z\rangle J_3X - \langle J_3X,Z\rangle J_3Y +\\
& 2\,\langle X,J_1Y\rangle J_1Z + 2\,\langle X,J_2Y\rangle J_2Z + 2\,\langle X,J_3Y\rangle J_3Z\Big].
\end{aligned}
$$

where $J_1,\ J_2,\ J_3$ are operators of quaternionic structure

$J_1J_2 = J_3,\ J_2J_3 = J_1,\ J_3J_1 = J_2,\ J_i^2 = -E,\ \langle X, J_iY\rangle = -\langle J_iX, Y\rangle,\ i = \overline{1,3}$

Introduce the unit sphere type operator S acting as

$$S(Z) \overset{def}{=} S_{XY}Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y,$$

the operators $S_i(Z)$ acting as

$$S_i(Z) \overset{def}{=} S_{J_iX\,J_iY}Z = \langle J_iY, Z\rangle J_iX - \langle J_iX, Z\rangle J_iY,\ i = \overline{1,3},$$

and the operator $\hat{S}(Z)$ acting as

$$\hat{S}(Z) \equiv S_1(Z) + S_2(Z) + S_3(Z).$$

Finally, denote $m_i = \langle X, J_iY\rangle,\ i = \overline{1,3},\ m^2 = m_1^2 + m_2^2 + m_3^2,\ \mathcal{J} = m_1J_1 + m_2J_2 + m_3J_3$. Then the curvature operator under consideration takes the form

$$R_{XY}Z = \frac{c}{4}\Big[S + \hat{S} + 2\mathcal{J}\Big]Z. \qquad (19)$$

Since $|X \wedge Y| = |(J_iX) \wedge (J_iY)|$ $(i = \overline{1,3})$, the operators S and S_i satisfy

$$S^3 = -b^2S, \quad S_i^3 = -b^2S_i\ (i = \overline{1,3}),$$

where $b^2 = |X \wedge Y|^2$.

The table of products for the operators S and $\hat{S}$ is the following one.

(20)

	S	S_1	S_2	S_3	J_1	J_2	J_3
S	S^2	$m_1J_1S_1$	$m_2J_2S_2$	$m_3J_3S_3$	J_1S_1	J_2S_2	J_3S_3
S_1	m_1J_1S	S_1^2	$-m_3J_3S_2$	$-m_2J_2S_3$	J_1S	J_2S_3	J_3S_2
S_2	m_2J_2S	$-m_3J_3S_1$	S_2^2	$-m_1J_1S_3$	J_1S_3	J_2S	J_3S_1
S_3	m_3J_3S	$-m_2J_2S_1$	$-m_1J_1S_2$	S_3^2	J_1S_2	J_2S_1	J_3S
J_1	S_1J_1	SJ_1	S_3J_1	S_2J_1	$-E$	J_3	$-J_2$
J_2	S_2J_2	S_3J_2	SJ_2	S_1J_2	$-J_3$	$-E$	J_1
J_3	S_3J_3	S_2J_3	S_1J_3	SJ_3	J_2	$-J_1$	$-E$

The expressions for products $SS_i,\ S_iS,\ SJ_i$ one can find similar to the table (16) making formal replacements $\hat{S} \to S_i$ and $J \to J_i$. As concerns the other entries, we have

$$
\begin{aligned}
&(S_1S_2)(Z) = \\
&\quad S_{J_1X\,J_1Y}S_{J_2X\,J_2Y}Z = S_{J_1X\,J_1Y}\Big[\langle J_2Y,Z\rangle J_2X - \langle J_2X,Z\rangle J_2Y\Big] = \\
&\quad \Big\langle J_1Y, \langle J_2Y,Z\rangle J_2X - \langle J_2X,Z\rangle J_2Y\Big\rangle J_1X - \\
&\quad \Big\langle J_1X, \langle J_2Y,Z\rangle J_2X - \langle J_2X,Z\rangle J_2Y\Big\rangle J_1Y = \\
&\quad \langle J_1Y,J_2X\rangle\langle J_2Y,Z\rangle J_1X + \langle J_1X,J_2Y\rangle\langle J_2X,Z\rangle J_1Y = \\
&\quad J_1\Big[\langle X,J_3Y\rangle\langle J_2Y,Z\rangle X - \langle X,J_3Y\rangle\langle J_2X,Z\rangle Y\Big] = \\
&\quad -J_1J_2\Big[m_3\langle J_2Y,Z\rangle J_2X - m_3\langle J_2X,Z\rangle J_2Y\Big] = (-m_3J_3S_2)(Z),
\end{aligned}
$$

$$
\begin{aligned}
(S_1J_1)(Z) = S_{J_1X\,J_1Y}J_1Z = \langle J_1Y, J_1Z\rangle J_1X - \langle J_1X, J_1Z\rangle J_1Y = \\
J_1[\langle Y,Z\rangle X - \langle X,Z\rangle Y] = (J_1S)(Z),
\end{aligned}
$$

$$
\begin{aligned}
&(S_1J_2)(Z) = S_{J_1X\,J_1Y}J_2Z = \langle J_1Y, J_2Z\rangle J_1X - \langle J_1X, J_2Z\rangle J_1Y = \\
&\quad J_1[\langle J_3Y,Z\rangle X - \langle J_3X,Z\rangle Y] = \\
&\qquad\qquad - J_1J_3[\langle J_3Y,Z\rangle J_3X - \langle J_3X,Z\rangle J_3Y] = (J_2S_3)(Z)
\end{aligned}
$$

and so on.

From (20) we see that

$$
\begin{aligned}
&(S+\hat{S})\mathcal{J} = (S+S_1+S_2+S_3)(m_1J_1+m_2J_2+m_3J_3) = \\
&m_1J_1S_1 + m_2J_2S_2 + m_3J_3S_3 + m_1J_1S + m_2J_2S_3 + m_3J_3S_2 + m_1J_1S_3 + \\
&m_2J_2S + m_3J_3S_1 + m_1J_1S_2 + m_2J_2S_1 + m_3J_3S = \\
&(m_1J_1+m_2J_2+m_3J_3)(S+S_1+S_2+S_3) = \mathcal{J}(S+\hat{S}).
\end{aligned}
$$

Therefore, the operators $(S+\hat{S})$ $\mathcal{J}$ commute and hence for the operator $R = \frac{c}{4}\{(S+\hat{S}) + 2\mathcal{J}\}$ the usual formula for powers can be applied:

$$
R^n = \left(\frac{c}{4}\right)^n \sum_{l=0}^{n} \binom{n}{l} \left(S+\hat{S}\right)^{n-l} 2^l \left(\mathcal{J}\right)^l .
$$

The powers for $\mathcal{J}$ can be found trivially, since

$$
\begin{aligned}
\mathcal{J}^2 = \quad & m_1^2J_1^2 + m_1m_2(J_1J_2+J_2J_1) + m_1m_3(J_1J_3+J_3J_1) + m_2^2J_2^2 + \\
& m_2m_3(J_2J_3+J_3J_2) + m_3^2J_3^2 = -m_1^2E - m_2^2E - m_3^2E = -m^2E,
\end{aligned}
$$

where $m_1^2+m_2^2+m_3^2 = m^2$.

As concerns the powers of $S+\hat{S}$, the following proposition gives the answer.

Proposition 3.1 *The operator $S+\hat{S}$ possesses the recurrent property*

$$
(S+\hat{S})^5 = -2\,(b^2+m^2)(S+\hat{S})^3 - (b^2-m^2)(S+\hat{S}),
$$

where $b^2 = |X\wedge Y|^2$ and $m^2 = m_1^2+m_2^2+m_3^2 = \langle X, J_1Y\rangle^2 + \langle X, J_2Y\rangle^2 + \langle X, J_3Y\rangle^2$.

Proof. The proof is technical and in what follows we will use some auxiliary operator products. Namely,

$$\begin{array}{ll} S\hat{S} = S\mathcal{J}, & \hat{S}S = \mathcal{J}S, \\ S\mathcal{J}S = -m^2 S, & S\hat{S}\mathcal{J} = -m^2 S, \\ S(S_1^2 + S_2^2 + S_3^2) = S^2\mathcal{J}, & \hat{S}S^2 = \mathcal{J}S^2, \\ \hat{S}S\mathcal{J} = \mathcal{J}S\mathcal{J}, & \hat{S}\mathcal{J}S = \mathcal{J}S^2, \\ \hat{S}^2 = S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + \mathcal{J}S, & \\ \hat{S}(S_1^2 + S_2^2 + S_3^2) = -b^2\hat{S} - (S_1^2 + S_2^2 + S_3^2)\mathcal{J} + \mathcal{J}S^2. & \end{array} \tag{21}$$

The proof is straightforward. Applying (20), we get
$S\hat{S} = S(S_1 + S_2 + S_3) = m_1 J_1 S_1 + m_2 J_2 S_2 + m_3 J_3 S_3 =$
$m_1 S J_1 + m_2 S J_2 + m_3 S J_3 = S\mathcal{J}$.

In a similar way we find
$\hat{S}S = (S_1 + S_2 + S_3)S = m_1 J_1 S + m_2 J_2 S + m_3 J_3 S = \mathcal{J}S$,
$\hat{S}^2 = (S_1 + S_2 + S_3)(S_1 + S_2 + S_3) = S_1^2 + S_2^2 + S_3^2 + S_1 S_2 + S_1 S_3 + S_2 S_1 + S_2 S_3 + S_3 S_1 + S_3 S_2 = S_1^2 + S_2^2 + S_3^2 - m_3 S_1 J_3 - m_2 S_1 J_2 - m_3 S_2 J_3 - m_1 S_2 J_2 - m_2 S_3 J_2 - m_1 S_3 J_1 = S_1^1 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + m_1 S_1 J_1 + m_2 S_2 J_2 + m_3 S_3 J_3 = S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + \mathcal{J}S$,

$S\mathcal{J}S = S(m_1 J_1 + m_2 J_2 + m_3 J_3)S = (m_1 J_1 S_1 + m_2 J_2 S_2 + m_3 J_3 S_3)S = -m_1^2 S - m_2^2 S - m_3^2 S = -m^2 S$,

$S\hat{S}\mathcal{J} = S\mathcal{J}\mathcal{J} = -m^2 S$,

$S(S_1^2 + S_2^2 + S_3^2) = m_1 S J_1 S_1 + m_2 S J_2 S_2 + m_3 S J_3 S_3 = m_1 S^2 J_1 + m_2 S^2 J_2 + m_3 S^2 J_3 = S^2\mathcal{J}$,

$\hat{S}\mathcal{J}S = (S_1 + S_2 + S_3)(m_1 J_1 + m_2 J_2 + m_3 J_3)S = (m_1 J_1 S + m_2 J_2 S_3 + m_3 J_3 S_2 + m_1 J_1 S_3 + m_2 J_2 S + m_3 J_3 S_1 + m_1 J_1 S_2 + m_2 J_2 S_1 + m_3 J_3 S)S = \mathcal{J}S^2 + m_2 J_2 m_3 J_3 S + m_3 J_3 m_2 J_2 S + m_1 J_1 m_3 J_3 S + m_3 J_3 m_1 J_1 S + m_1 J_1 m_2 J_2 S + m_2 J_2 m_1 J_1 S = \mathcal{J}S^2$,

$\hat{S}(S_1^2 + S_2^2 + S_3^2) = (S_1 + S_2 + S_3)(S_1^2 + S_2^2 + S_3^2) = S_1^3 + S_2^3 + S_3^3 + S_1 S_2^2 + S_1 S_3^2 + S_2 S_1^2 + S_2 S_3^2 + S_3 S_1^2 + S_3 S_2^2 = -b^2\hat{S} - m_3 J_3 S_2^2 - m_2 J_2 S_3^2 - m_3 J_3 S_1^2 - m_1 J_1 S_3^2 - m_2 J_2 S_1^2 - m_1 J_1 S_2^2 = -b^2\hat{S} - m_3 S_1^2 J_3 - m_2 S_1^2 J_2 - m_3 S_2^2 J_3 - m_1 S_2^2 J_1 - m_2 S_3^2 J_2 - m_1 S_3^2 J_1 = -b^2\hat{S} - (S_1^2 + S_2^2 + S_3^2)\mathcal{J} + m_1 S_1^2 J_1 + m_2 S_2^2 J_2 + m_3 S_3^2 J_3 = -b^2\hat{S} + (S_1^2 + S_2^2 + S_3^2)\mathcal{J} + m_1 J_1 S^2 + m_2 J_2 S^2 + m_3 J_3 S^2 =$
$-b^2\hat{S} + (S_1^2 + S_2^2 + S_3^2)\mathcal{J} + \mathcal{J}S^2$.

Now we are ready to find the powers of $(S + \hat{S})$. Using (21), we get
$(S + \hat{S})^2 = S^2 + S\hat{S} + \hat{S}S + \hat{S}^2 = S^2 + S\mathcal{J} + \mathcal{J}S + S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + \mathcal{J}S =$
$S^2 + S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J} + S_1^2 + S_2^2 + S_3^2$.

Multiplying the result by $S + \hat{S}$ and applying again (21), we find

$(S+\hat{S})^3 = (S+\hat{S})[S^2+S\mathcal{J}+2\mathcal{J}S-\hat{S}\mathcal{J}+S_1^2+S_2^2+S_3^2] = S^3+S^2\mathcal{J}+2S\mathcal{J}S-S\hat{S}\mathcal{J}+S(S_1^2+S_2^2+S_3^2)+\hat{S}S^2+\hat{S}S\mathcal{J}+2\hat{S}\mathcal{J}S-\hat{S}^2\mathcal{J}+\hat{S}(S_1^2+S_2^2+S_3^2) = -b^2S+S^2\mathcal{J}-2m^2S+m^2S+S^2\mathcal{J}+\mathcal{J}S^2+\mathcal{J}S\mathcal{J}+2\hat{S}\mathcal{J}S-[(S_1^2+S_2^2+S_3^2)-\hat{S}\mathcal{J}+\mathcal{J}S]\mathcal{J}+[-b^2\hat{S}-(S_1^2+S_2^2+S_3^2)\mathcal{J}+\mathcal{J}S^2] = -(b^2+m^2)S+2S^2\mathcal{J}+\mathcal{J}S^2+\mathcal{J}S\mathcal{J}+2\mathcal{J}S^2-(S_1^2+S_2^2+S_3^2)\mathcal{J}+\hat{S}\mathcal{J}^2-\mathcal{J}S\mathcal{J}-b^2\hat{S}-(S_1^2+S_2^2+S_3^2)\mathcal{J}+\mathcal{J}S^2 = -(b^2+m^2)S+2S^2\mathcal{J}+4\mathcal{J}S^2-2(S_1^2+S_2^2+S_3^2)\mathcal{J}-m^2\hat{S}-b^2\hat{S} =$

$$-(b^2+m^2)(S+\hat{S})+2S^2\mathcal{J}+4\mathcal{J}S^2-2(S_1^2+S_2^2+S_3^2)\mathcal{J}.$$

Continue the process.

$(S+\hat{S})^4 = (S+\hat{S})[-(b^2+m^2)(S+\hat{S})+2S^2\mathcal{J}+4\mathcal{J}S^2-2(S_1^2+S_2^2+S_3^2)\mathcal{J}] = -(b^2+m^2)(S+\hat{S})^2+2S^3\mathcal{J}+4S\mathcal{J}S^2-2S(S_1^2+S_2^2+S_3^2)\mathcal{J}+2\hat{S}S^2\mathcal{J}+4\hat{S}\mathcal{J}S^2-2\hat{S}(S_1^2+S_2^2+S_3^2)\mathcal{J} = -(b^2+m^2)(S+\hat{S})^2-2b^2S\mathcal{J}-4m^2S^2-2S^2\mathcal{J}\mathcal{J}+2\mathcal{J}S^2\mathcal{J}+4\mathcal{J}S^3-2[-b^2\hat{S}-(S_1^2+S_2^2+S_3^2)\mathcal{J}+\mathcal{J}S^2]\mathcal{J} = -(b^2+m^2)(S+\hat{S})^2-2b^2S\mathcal{J}-4m^2S^2+2m^2S^2+2\mathcal{J}S^2\mathcal{J}-4b^2\mathcal{J}S+2b^2\hat{S}\mathcal{J}+2(S_1^2+S_2^2+S_3^2)\mathcal{J}^2-2\mathcal{J}S^2\mathcal{J} = -(b^2+m^2)(S+\hat{S})^2-2b^2S\mathcal{J}-2m^2S^2-4b^2\mathcal{J}S+2b^2\hat{S}\mathcal{J}-2m^2(S_1^2+S_2^2+S_3^2) = -(b^2+m^2)(S+\hat{S})^2-2m^2[S^2+S\mathcal{J}+2\mathcal{J}S-\hat{S}\mathcal{J}+(S_1^2+S_2^2+S_3^2)]+(2m^2-2b^2)(S\mathcal{J}+2\mathcal{J}S-\hat{S}\mathcal{J}) =$

$$-(b^2+3m^2)(S+\hat{S})^2+(2m^2-2b^2)(S\mathcal{J}+2\mathcal{J}S-\hat{S}\mathcal{J}).$$

Finally,

$(S+\hat{S})^5 = (S+\hat{S})[-(b^2+3m^2)(S+\hat{S})^2+(2m^2-2b^2)(S\mathcal{J}+2\mathcal{J}S-\hat{S}\mathcal{J})] = -(b^2+3m^2)(S+\hat{S})^3+(2m^2-2b^2)[S^2\mathcal{J}+2S\mathcal{J}S-S\hat{S}\mathcal{J}+\hat{S}S\mathcal{J}+2\hat{S}\mathcal{J}S-\hat{S}^2\mathcal{J}] = -(b^2+3m^2)(S+\hat{S})^3+(2m^2-2b^2)[S^2\mathcal{J}-2m^2S-S\mathcal{J}^2+\mathcal{J}S\mathcal{J}+2\mathcal{J}S^2-(S_1^2+S_2^2+S_3^2-\hat{S}\mathcal{J}+\mathcal{J}S)\mathcal{J}] = -(b^2+3m^2)(S+\hat{S})^3+(2m^2-2b^2)[S^2\mathcal{J}-m^2S+\mathcal{J}S\mathcal{J}+2\mathcal{J}S^2-(S_1^2+S_2^2+S_3^2)\mathcal{J}+\hat{S}\mathcal{J}^2-\mathcal{J}S\mathcal{J}] = -(b^2+3m^2)(S+\hat{S})^3+(m^2-b^2)[2S^2\mathcal{J}+4\mathcal{J}S^2-2(S_1^2+S_2^2+S_3^2)\mathcal{J}-2m^2S-2m^2\hat{S}] = -(b^2+3m^2)(S+\hat{S})^3+(m^2-b^2)[2S^2\mathcal{J}+4\mathcal{J}S^2-2(S_1^2+S_2^2+S_3^2)\mathcal{J}-(m^2+b^2)(S+\hat{S})+(b^2-m^2)(S+\hat{S})] = -(b^2+3m^2)(S+\hat{S})^3+(m^2-b^2)[(S+\hat{S})^3+(b^2-m^2)(S+\hat{S})] =$

$$-2(b^2+m^2)(S+\hat{S})^3-(b^2-m^2)^2(S+\hat{S})$$

which completes the proof.

■

Thus,

$$(S+\hat{S})^5 = \mathrm{Lin}\,((S+\hat{S})^3, S+\hat{S}) \tag{22}$$

On the other hand, setting for brevity $R_{XY} = R$, from (19) we derive

$$S+\hat{S} = \frac{4}{c}R-2\mathcal{J} = \mathrm{Lin}\,(R,\mathcal{J}) \tag{23}$$

Since $(S+\hat{S})$ and $\mathcal{J}$ commute, (19) implies the commutation of R and $\mathcal{J}$. Keeping this and $\mathcal{J}^2 = -m^2E$, from (23) we derive

$$(S+\hat{S})^3 = \left(\frac{4}{c}\right)^3 R^3 + \mathrm{Lin}\,(\mathcal{J}R^2, R, \mathcal{J}) \tag{24}$$

$$(S+\hat{S})^5 = \left(\frac{4}{c}\right)^5 R^5 + \mathrm{Lin}\,(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J}) \tag{25}$$

From (22), (23) and (24)

$$(S+\hat{S})^5 = \mathrm{Lin}\ \Big[\mathrm{Lin}\,(R^3, \mathcal{J}R^2, R, \mathcal{J}), \mathrm{Lin}\,(R, \mathcal{J})\Big] = \mathrm{Lin}\,(R^3, \mathcal{J}R^2, R, \mathcal{J}).$$

So, finally from (25)

$$R^5 = \mathrm{Lin}\,(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J}).$$

It is easy to trace that the coefficients of all linear combinations are polynomials in $\frac{1}{c}, b, m$. To complete the proof we should remark that
$R^6 = R^5R = \mathrm{Lin}\,(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J})R = \mathrm{Lin}\,(\mathcal{J}R^5, R^4, \mathcal{J}R^3, R^2, \mathcal{J}R) =$
$\mathrm{Lin}\ \Big[\mathcal{J}\,\mathrm{Lin}\,(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J}), R^4, \mathcal{J}R^3, R^2, \mathcal{J}R\Big] =$
$\mathrm{Lin}\,(R^4, \mathcal{J}R^3, R^2, \mathcal{J}R, E)$
which allows to find all powers of R inductively.

■

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