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On Affine Immersions with Flat Connections

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In this paper, the multidimensional affine immersions with flat connections of maximal pointwise codimension are studied. The estimates on the dimensions of kernel and the image of the shape (Weingarten) operator and the affine fundamental form are obtained. Some properties of nullity distributions on the immersed submanifold are considered and the examples of affine immersions of high codimension with flat connection are given.

Key words: affine immersion, flat connection, pointwise codimension, nullity.

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Introduction

Let M^n be an affine n-dimensional manifold with connection ∇. Denote by $\mathbb{R}^{n+k}$ a standard affine space with flat connection D. According to [7], an immersion $f : (M^n, \nabla) \to (\mathbb{R}^{n+k}, D)$ is said to be affine if along f there exists a k-dimensional transversal differentiable distribution Q such that the affine Gauss decomposition

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,\ Y),$$

where $f_*(\nabla_X Y) \in TM^n$ and $h(X,\ Y) \in Q$, holds. The component $h(X,\ Y)$ is called *the affine fundamental form*. The affine fundamental form defines a mapping

$$h : T_x M^n \times T_x M^n \to Q_x.$$

The rank of this mapping is called the *rank of the affine fundamental form* at $x \in M^n$ or the *pointwise codimension* of the affine immersion.

If $\xi \in Q$, then the affine Weingarten decomposition

$$D_X \xi = -f_*(S_\xi X) + \nabla^\perp_X \xi,$$

where $f_*(S_\xi X) \in TM^n$ and $\nabla^\perp_X \xi \in Q$, holds. This decomposition defines *the shape operator* S_ξ and *the transversal connection* $\nabla^\perp$.

Denote by $\xi_1, \dots \xi_k$ the linearly independent vector fields in Q (a transversal affine frame). Then

$$D_X f_*(Y) = f_*(\nabla_X Y) + h^\alpha(X,\ Y)\xi_\alpha, \tag{1}$$

$$D_X \xi_\alpha = -f_*(S_\alpha X) + \tau_\alpha^\beta(X)\xi_\beta. \tag{2}$$

The null space or *kernel* $\mathcal{N}_x$ of the affine fundamental form at $x \in M$ is defined by

$$\mathcal{N}_x = \ker h_x := \bigcap_{\alpha=1}^{k} \ker h_x^\alpha,$$

where $\ker h_x^\alpha = \{X \in T_xM : h^\alpha(X,Y) = 0 \text{ for all } Y \in T_xM\}$. The *nullity index* of the affine fundamental form at a point x is defined by $\mu_x = \dim \mathcal{N}_x$. Neither $\mathcal{N}_x$ nor pointwise codimension depends on a choice of transversal distribution (see (8)).

The *null space* or *kernel* of the shape operator $\mathcal{S}_x$ is defined by

$$\mathcal{S}_x = \ker S_x := \bigcap_{\alpha=1}^{k} (\ker S_\alpha)_x,$$

where $(\ker S_\alpha)_x = \{X \in T_xM : S_\alpha X = 0\}$.

The subspaces $\{\mathcal{N}_x\}_{x\in M^n}$ form a smooth distribution on M^n called *the nullity distribution of the affine fundamental form*. The subspaces $\{\mathcal{S}_x\}_{x\in M^n}$ form a smooth distribution which we call *the nullity distribution of the shape operator*.

The following *Hartman-Nirenberg* theorem is well known for hypersurfaces in Euclidean space [4].

Let $f : M^n \to E^{n+1}$ be a complete connected C^2-smooth orientable hypersurface. If f is of constant zero curvature, then it is an $(n-1)$-dimensional cylinder.

The generalization of this theorem to higher codimensions and various Riemannian spaces has a long history. The case of submanifold with nullity distribution in generic Riemannian manifold was considered by A. Borisenko [1]. In spite of evident affine nature of the Hartmann-Nirenberg theorem, a progress in its generalization to higher codimensions is not much impressive probably because the nullity distributions of the second fundamental form and the shape operator are different in general.

Complete affine hypersurfaces with flat connections in the standard affine $\mathbb{R}^{n+1}$ were studied by K. Nomizu and U. Pinkall [6] (see also [7]). They classified immersions as

(i) a hyperplane ($h \equiv 0$);

(ii) a graph ($S \equiv 0$);

(iii) an affine cylinder with $(n-1)$-dimensional rulings ($h \neq 0,\ S \neq 0$).

In the last case they proved that $\dim \operatorname{im} S = 1$ and the nullity of the affine fundamental form coincide with the nullity of the shape operator, $\dim \mathcal{S} = \dim \mathcal{N} = n-1$. The nullity distribution $\mathcal{N}$ is integrable and the nullity foliation $\mathcal{FN}$ is totally geodesic in $\mathbb{R}^{n+1}$.

B. Opozda [8] proved that cylinders are the only affine hypersurfaces of 1-codimensional nullity which admit a non-flat locally symmetric induced connection.

S.S. Chern and N.H. Kuiper [3] proved that the nullity index satisfies $\mu \geq n-k$ in the case of immersion $f : M^n \to \mathbb{E}^{n+k}$ $(k < n)$ of constant zero curvature.

In full similarity to a hypersurface treatment, in the case of *multi-codimensional affine immersion* $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ it is easy to see that

(i) if $h \equiv 0$, then f is a totally geodesic immersion, and $f(\mathbb{R}^n)$ is an n-dimensional affine subspace;

(ii) if $S \equiv 0$, then $f(\mathbb{R}^n)$ is a graph.

In the cases of $S \not\equiv 0$ and $h \not\equiv 0$, we obtain the following estimations.

Theorem 1. *Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+k}, D)$ be an affine immersion of codimension $k < n$ with maximal pointwise codimension and flat connection ∇. Then*

(1) $\dim \ker S \geq n-k$;

(2) $\ker h \subseteq \ker S$;

(3) $\dim \operatorname{im} S \leq k$;

(4) *if* $\dim \operatorname{im} S = k$ *then* $\dim \ker S = n-k$ *and* $\ker h = \ker S$.

In Euclidean case ($\mu = \text{const} \neq 0$), it is known that the nullity distribution is integrable and totally geodesic [5, 1], the normal space is stationary along the leaves [1]. In affine case we obtain the similar result.

Theorem 2. *Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+k}, D)$ be an affine immersion such that* $\dim \ker h = \text{const} \neq 0,\ \dim \ker S = \text{const} \neq 0$. *Then*

(1) *the nullity distribution $\mathcal{S}$ of the shape operator is integrable on M^n;*

(2) *the nullity distribution $\mathcal{N}$ of the affine fundamental form is integrable, the leaves are totally geodesic in $\mathbb{R}^{n+k}$;*

(3) *there exists a transversal distribution which is stationary along the leaves of the foliation $\mathcal{FN}$;*

(4) *if (M^n, ∇) is complete, then each leaf of the foliation $\mathcal{FN}$ is complete.*

Corollary 2 *Let $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ $(k < n)$ be an affine immersion with* $\dim \operatorname{im} S = k$*. Then the submanifold $f(\mathbb{R}^n)$ is foliated by the $(n-k)$-dimensional affine subspaces. The transversal distribution is stationary along the subspaces.*

A maximality of $\dim \operatorname{im} S$ is essential. Otherwise (see Example 1) an affine immersion with flat connection could have a nontrivial S and a trivial kernel of the affine fundamental form but not the rectilinear rulings. Unlike hypersurfaces, there exist affine immersions $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ $(k \geq 2)$ such that the immersed submanifold is foliated by the $(n-k)$-dimensional affine subspaces, is of maximal pointwise codimension ($\dim \operatorname{im} S = k$), but is not a cylinder (see Example 2).

1. Preliminaries

Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+k}, D)$ be an affine immersion. The following basic equations are well known (see [6], [7]):
the affine Gauss equation

$$R(X,\ Y)Z = h^\alpha(Y,\ Z)S_\alpha X - h^\alpha(X,\ Z)S_\alpha Y; \tag{3}$$

the affine Codazzi equations for h

$$(\nabla_X h^\alpha)(Y,\ Z) + \tau^\alpha_\beta(X)h^\beta(Y,\ Z) = (\nabla_Y h^\alpha)(X,\ Z) + \tau^\alpha_\beta(Y)h^\beta(X,\ Z); \tag{4}$$

the affine Codazzi equations for S

$$(\nabla_X S_\alpha)Y - \tau^\beta_\alpha(X)S_\beta Y = (\nabla_Y S_\alpha)X - \tau^\beta_\alpha(Y)S_\beta X; \tag{5}$$

the affine Ricci equations

$$\begin{aligned} h^\beta(X,\ S_\alpha Y) - h^\beta(Y,\ S_\alpha X) = X(\tau^\beta_\alpha(Y)) + \tau^\beta_\gamma(X)\tau^\gamma_\alpha(Y) \\ - Y(\tau^\beta_\alpha(X)) - \tau^\beta_\gamma(Y)\tau^\gamma_\alpha(X) - \tau^\beta_\alpha([X,\ Y]). \end{aligned} \tag{6}$$

The components of transversal connection, the affine fundamental form and the shape operator depend on a choice of transversal distribution and transversal frame.

Lemma 1. *Let M^n be a submanifold in $\mathbb{R}^{n+k}$ with transversal distribution $Q = \operatorname{span}\{\xi_1, \ldots,\ \xi_k\}$. Let $\bar{Q} = \operatorname{span}\{\bar{\xi}_1, \ldots,\ \bar{\xi}_k\}$ be a transformation of Q by*

$$\bar{\xi}_\alpha = \Phi^\beta_\alpha \xi_\beta + Z_\alpha, \tag{7}$$

where Z_α are tangent vector fields on M^n, and $\Phi = [\Phi^\beta_\alpha]_{k\times k}$ is a nondegenerate matrix with smooth entries. Then the components of the affine fundamental form, *the induced connection, the shape operators and the transversal connection forms change as follows:*

$$\bar{h}^{\alpha}(X,\ Y) = [\Phi^{-1}]^{\alpha}_{\beta} h^{\beta}(X,\ Y) \tag{8}$$

$$\bar{\nabla}_X Y = \nabla_X Y - [\Phi^{-1}]^{\alpha}_{\beta} h^{\beta}(X,\ Y) Z_{\alpha} \tag{9}$$

$$\bar{S}_{\alpha} X = \Phi^{\beta}_{\alpha} S_{\beta} X - \nabla_X Z_{\alpha} + \bar{\tau}^{\beta}_{\alpha}(X) Z_{\beta} \tag{10}$$

$$\bar{\tau}^{\beta}_{\alpha}(X) = [\Phi^{-1}]^{\beta}_{\gamma}\{\tau^{\gamma}_{\delta}(X)\Phi^{\delta}_{\alpha} + h^{\gamma}(X,\ Z_{\alpha}) + X(\Phi^{\gamma}_{\alpha})\} \tag{11}$$

P r o o f. From (1) and (7) we derive

$$D_X Y = \nabla_X Y + h^{\alpha}(X,\ Y)\xi_{\alpha} = \bar{\nabla}_X Y + \bar{h}^{\alpha}(X,\ Y)\bar{\xi}_{\alpha} = \bar{\nabla}_X Y + \bar{h}^{\alpha}(X,\ Y)(\Phi^{\beta}_{\alpha}\xi_{\beta} + Z_{\alpha}).$$

Thus, we obtain (8) and (9).

Using (2) and (7), we have

$$D_X \bar{\xi}_{\alpha} = D_X(\Phi^{\beta}_{\alpha}\xi_{\beta} + Z_{\alpha}) = X(\Phi^{\beta}_{\alpha})\xi_{\beta} + \Phi^{\beta}_{\alpha}(-S_{\beta}X + \tau^{\gamma}_{\beta}(X)\xi_{\gamma}) + \nabla_X Z_{\alpha} + h^{\gamma}(X,\ Z_{\alpha})\xi_{\gamma} = \nabla_X Z_{\alpha} - \Phi^{\beta}_{\alpha} S_{\beta} X + \{\Phi^{\beta}_{\alpha}\tau^{\gamma}_{\beta}(X) + h^{\gamma}(X,\ Z_{\alpha}) + X(\Phi^{\gamma}_{\alpha})\}\xi_{\gamma}.$$

On the other hand,

$$D_X \bar{\xi}_{\alpha} = -\bar{S}_{\alpha} X + \bar{\tau}^{\beta}_{\alpha}(X)\bar{\xi}_{\beta} = -\bar{S}_{\alpha} X + \bar{\tau}^{\beta}_{\alpha}(X)\{\Phi^{\gamma}_{\beta}\xi_{\gamma} + Z_{\beta}\} = -\bar{S}_{\alpha} X + \bar{\tau}^{\beta}_{\alpha}(X) Z_{\beta} + \bar{\tau}^{\beta}_{\alpha}(X)\Phi^{\gamma}_{\beta}\xi_{\gamma}.$$

Comparing the tangential and transversal components, we have (10) and (11). ∎

From (8), the rank of the affine fundamental form $h(X,\ Y) : T_x(M) \times T_x(M) \to Q_x$ does not depend on a choice of transversal distribution, and the following definition of the *pointwise codimension* of affine immersion is well- defined.

Definition 1. *The rank of the affine fundamental form is called the pointwise codimension of affine immersion.*

In fact, a pointwise codimension of affine immersion is nothing else but a dimension of the first normal space and it is the same as in the Euclidean case.

Let $\{e_1,\ e_2, \ldots,\ e_n\}$ be a tangent affine frame on M^n. Introduce the matrix

$$H(X) = \begin{pmatrix} h^1(X,\ e_1) & h^1(X,\ e_2) & \ldots & h^1(X,\ e_n) \\ h^2(X,\ e_1) & h^2(X,\ e_2) & \ldots & h^2(X,\ e_n) \\ \vdots & \vdots & \ddots & \vdots \\ h^k(X,\ e_1) & h^k(X,\ e_2) & \ldots & h^k(X,\ e_n) \end{pmatrix}_{k\times n}. \tag{12}$$

By (8), the change of transversal distribution by (7) implies

$$\bar{H}(X) = \Phi^{-1}H(X). \tag{13}$$

The maximal rank of $H(X)$ is equal to the pointwise codimension of affine immersion.

Let the affine immersion $f : (M^n,\ \nabla) \to (\mathbb{R}^{n+k},\ D)$ be given. To what extent is a transversal distribution defined?

Lemma 2. *Let $f : (M^n,\ \nabla) \to (\mathbb{R}^{n+k},\ D)$ be an affine immersion. Then the transversal distribution Q is determined uniquely up to the subdistribution $Q\setminus Q_1$, where Q_1 contains the image of the affine fundamental form.*

P r o o f. Assume that the rank of the affine fundamental form is equal to q. Let $Q(x)$ be some transversal distribution. Denote by $Q_1(x)$ a subdistribution of $Q(x)$ which contains the image of h_x at each $x \in M^n$. Then $\dim Q_1(x) = q$. Take a basis of $Q(x)$ in such a way that the first q vectors form a basis of $Q_1(x)$. Then $h^\alpha(X,\ Y) \equiv 0 \ \ (\alpha = \overline{q+1,k})$.

By (9), the induced connection will not change under the transformation (7) if

$$[\Phi^{-1}]^\alpha_\beta h^\beta(X,\ Y)Z_\alpha = 0.$$

Let $\{e_1,\ e_2, \ldots, e_n\}$ be an affine tangent frame on M^n. Then $Z_\alpha = z^i_\alpha e_i$, and we have

$$\begin{aligned} [\Phi^{-1}]^\alpha_\beta h^\beta(X,\ Y) z^i_\alpha e_i &= 0 \\ z^i_\alpha [\Phi^{-1}]^\alpha_\beta h^\beta(X,\ Y) &= 0 \\ (z^i_1,\ z^i_2, \ldots z^i_k) \cdot \Phi^{-1} \cdot \begin{pmatrix} h^1(X,\ Y) \\ h^2(X,\ Y) \\ \vdots \\ h^q(X,\ Y) \\ 0 \\ \vdots \\ 0 \end{pmatrix} &= 0 \qquad \text{for all } X,\ Y. \end{aligned} \tag{14}$$

Since $h^\alpha(X,\ Y) \equiv 0 \ \ (\alpha = \overline{q+1,k})$, the last $k-q$ rows of $H(X)$ consist of zeroes. Let X be such that the rank $H(X) = q$. In a capacity of Y, take those $e_{i_1}, \ldots, e_{i_q}$ for which the corresponding columns in $H(X)$ are linearly independent. Then (14) produces a homogeneous system of q linear equations in q variables $z^i_1,\ z^i_2, \ldots, z^i_q$ with nondegenerate matrix. This system has only the trivial solution. The other variables $z^i_{q+1}, \ldots, z^i_n$ can be taken arbitrarily.

If $q = k$, then the transversal distribution is determined uniquely.

Suppose $q < k$. Take the transversal frame as above. The transformation

$$\bar{\xi}_\alpha = \Phi_\alpha^\beta \xi_\beta + Z_\alpha,\ \Phi_{k\times k} = \begin{pmatrix} \Psi_{q\times q} & * \\ 0 & \Omega_{(k-q)\times(k-q)} \end{pmatrix},\ Z_\alpha \equiv 0\ (\alpha = \overline{1,q}),$$

where Ψ and Ω are some non-degenerate matrices, does not change the affine connection and maps Q_1 onto Q_1. Therefore, the transversal distribution is determined uniquely up to the subdistribution $Q(x) \setminus Q_1(x)$. ∎

Corollary 1. *If a pointwise codimension of affine immersion is maximal, then* $\operatorname{im} S$ *and* $\ker S$ *do not depend on a choice of transversal frame.*

The proof follows from (10).

Denote by $R^\perp(X,Y)\xi$ a curvature tensor of transversal connection. The Ricci equation (6) can be written as

$$R^\perp(X,Y)\xi = h(X, S_\xi Y) - h(Y, S_\xi X),$$

and hence if $X, Y \in (\ker h \cup \ker S)$, then $R^\perp(X,Y)\xi = 0$. Therefore, there exists a basis of transversal distribution such that

$$\tau_\alpha^\beta(X) = 0 \quad \text{for} \quad \alpha,\beta = \overline{1,k} \quad \text{and} \quad X \in (\ker h \cup \ker S). \tag{15}$$

2. Proofs of Theorems

In the case of multicodimensional affine immersion with flat connection, using the same arguments as in the case of hypersurface [6, 7], one can easily prove that (a) if $h \equiv 0$, then f is a totally geodesic immersion, and $f(\mathbb{R}^n)$ is an n-dimensional affine subspace; (b) if $S \equiv 0$, then $f(\mathbb{R}^n)$ is a graph immersion. The second case is presented separately as the following assertion.

Lemma 3. *Let* $f : (M^n,\ \nabla) \to (\mathbb{R}^{n+k},\ D)$ *be an affine immersion such that* $S \equiv 0$. *Then* f *is affinely equivalent to the graph immersion of some smooth map* $F : M^n \to \mathbb{R}^k$, *i.e.,*

$$f(u^1, \ldots,\ u^n) = (u^1, \ldots,\ u^n,\ F^1(u^1, \ldots,\ u^n), \ldots,\ F^k(u^1, \ldots,\ u^n)).$$

R e m a r k. In the case when the shape operator is zero, the Gauss equation (3) implies that ∇ is a flat connection.

Another example of the submanifold with flat connection provides *an affine cylinder with* $(n-r)$*-dimensional rulings*, i.e., a submanifold in $\mathbb{R}^{n+k}$ generated by a family of parallel $(n-r)$-dimensional affine subspaces which is parameterized by points of an r-dimensional submanifold (the so-called base of a cylinder).

Similarly to the Euclidean case (see [1], [2]), *a pointwise rank of the affine fundamental form* is defined by

$$r(x) = \max_{\xi \in Q_x} r(x, \xi),$$

where $r(x, \xi)$ is a rank of the affine fundamental form with respect to $\xi \in Q_x$.

P r o o f of the Theorem 1. Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+k}, D)$ $(k < n)$ be an affine immersion with maximal pointwise codimension and pointwise rank r $(r \leq n)$ of the affine fundamental form. Since ∇ is a flat connection, the Gauss equation (3) implies

$$h^\alpha(Y,\ Z) S_\alpha X = h^\alpha(X,\ Z) S_\alpha Y. \tag{16}$$

Since the pointwise rank of the affine fundamental form is equal to r, we have $\dim \ker h = n - r \geq 0$. Choose the tangent frame such that $\ker h = \operatorname{span}\{e_{r+1}, \ldots,\ e_n\}$.

Suppose $r < k$. Substituting $Y = e_j$ $(j = \overline{r+1, n})$ into the Gauss equation (16), we get

$$h^\alpha(X, Z) S_\alpha e_j = 0$$

for all X, Z. As the rank of the affine fundamental form is equal to k, for the arbitrary fixed $\alpha = \alpha_0$ there exists X and Z such that

$$h^{\alpha_0}(X, Z) = 1,\ h^\alpha(X, Z) = 0 \quad \alpha \neq \alpha_0.$$

Hence, $S_{\alpha_0} e_j = 0$. Thus, $\ker h \subseteq \ker S$ and $\dim \ker S \geq n - r \geq n - k$.

In the case $r \geq k$, we have

$$H(X) = \begin{pmatrix} h^1(X,\ e_1) & h^1(X,\ e_2) & \ldots & h^1(X,\ e_r) & 0 & \ldots & 0 \\ h^2(X,\ e_1) & h^2(X,\ e_2) & \ldots & h^2(X,\ e_r) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h^k(X,\ e_l) & h^k(X,\ e_2) & \ldots & h^k(X,\ e_r) & 0 & \ldots & 0 \end{pmatrix}.$$

We can assume that e_1 is such that $\operatorname{rg} H(e_1) = k$. Applying a transformation $\bar{\xi}_\alpha = \Phi_\alpha^\beta \xi_\beta$ $(\det \Phi_{k\times k} \neq 0)$ and, if necessary, renumbering the vectors of the tangent frame, we can reduce the matrix $H(e_1)$ to the form

$$H(e_1) = \begin{pmatrix} 1 & 0 & \ldots & 0 & h^1(e_1,\ e_{k+1}) & \ldots & h^1(e_1,\ e_r) & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & h^2(e_1,\ e_{k+1}) & \ldots & h^2(e_1,\ e_r) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & h^k(e_1,\ e_{k+1}) & \ldots & h^k(e_1,\ e_r) & 0 & \ldots & 0 \end{pmatrix}.$$

There exists X such that $h^\alpha(e_1,\ X)=0$ for all $\alpha=\overline{1,k}$. Indeed, we get the following system of equations on X:

$$\begin{cases} x_1+x_{k+1}h^1(e_1,\ e_{k+1})+\ldots+x_rh^1(e_1,\ e_r) &= & 0\\ x_2+x_{k+1}h^2(e_1,\ e_{k+1})+\ldots+x_rh^2(e_1,\ e_r) &= & 0\\ \vdots & & \vdots\\ x_k+x_{k+1}h^k(e_1,\ e_{k+1})+\ldots+x_rh^k(e_1,\ e_r) &= & 0. \end{cases}$$

This system has the $r-k$ linearly independent solutions which can be taken as the vectors of a new affine frame $\tilde{e}_j$ for $j=\overline{k+1,r}$. In new frame

$$\tilde{e}_i=e_i\ (i=\overline{1,k},\ \overline{r+1,n}),\quad \tilde{e}_j=-\sum_{s=1}^{k}h^s(e_1,\ e_j)e_s+e_j\ (j=\overline{k+1,r}).$$

The entries of $H(\tilde{e}_1)$ take the form

$$h^\alpha(\tilde{e}_1,\ \tilde{e}_i)=\delta_i^\alpha,\ h^\alpha(\tilde{e}_1,\ \tilde{e}_j)=0,\ i=\overline{1,k},\ j=\overline{k+1,n},\ \alpha=\overline{1,k}.$$

Briefly, we can write

$$H_{k\times n}(\tilde{e}_1)=\begin{pmatrix} E_{k\times k} & O_{k\times(n-k)}\end{pmatrix}.$$

By substituting $X=\tilde{e}_i,\ Y=\tilde{e}_j,\ Z=\tilde{e}_1$ into the Gauss equation (16), we obtain

$$h^\alpha(\tilde{e}_j,\ \tilde{e}_1)S_\alpha\tilde{e}_i=h^\alpha(\tilde{e}_i,\ \tilde{e}_1)S_\alpha\tilde{e}_j. \tag{17}$$

For $i=\overline{1,k},\ j=\overline{k+1,n}$ we have

$$0\cdot S_\alpha\tilde{e}_i=\delta_i^\alpha\cdot S_\alpha\tilde{e}_j$$

$$S_\alpha\tilde{e}_j=0\ j=\overline{k+1,n},\ \alpha=\overline{1,k}.$$

Thus, $\operatorname{span}\{\tilde{e}_{k+1},\ldots,\ \tilde{e}_n\}\subseteq\ker S$. Therefore, $\dim\ker S\geq n-k$ and $\ker h\subseteq\ker S$. (Observe that in fact we obtained a more exact inequality, namely $\dim\ker S\geq n-\min\{r,k\}$.)

From (17), for $i,j=\overline{1,k}$ we get

$$S_j\tilde{e}_i=h^\alpha(\tilde{e}_i,\ \tilde{e}_j)S_\alpha\tilde{e}_1. \tag{18}$$

Therefore, $\dim\operatorname{im}\ S\leq k$ and $\operatorname{im}\ S=\operatorname{span}\{S_1\tilde{e}_1,\ldots,S_k\tilde{e}_1\}$. The third statement of the theorem is proved.

If $\dim\operatorname{im}\ S=k$, then $S_1\tilde{e}_1,\ldots,S_k\tilde{e}_1$ are linearly independent and form a basis of the image of S. From (18), for $j=1,\ i=\overline{1,k}$ we obtain

$$S_1\tilde{e}_i=S_i\tilde{e}_1\neq 0.$$

Therefore, $\tilde{e}_i \notin \ker S$ for all $i = \overline{1,k}$. Hence, $\mathrm{span}\{\tilde{e}_{k+1}, \dots, \tilde{e}_n\} = \ker S$ and $\dim \ker S = n - k$.

Consider (16) for $j = \overline{k+1,n}$ and $i = \overline{1,n}$. Then

$$h^\alpha(\tilde{e}_1, \tilde{e}_j)S_\alpha\tilde{e}_i = h^\alpha(\tilde{e}_i, \tilde{e}_j)S_\alpha\tilde{e}_1,$$

and we see that

$$0 = h^\alpha(\tilde{e}_i, \tilde{e}_j)S_\alpha\tilde{e}_1.$$

Since $S_1\tilde{e}_1, \dots, S_k\tilde{e}_1$ are linearly independent, then $h^\alpha(\tilde{e}_i, \tilde{e}_j) = 0$ for all $j = \overline{k+1,n}$ and $i = \overline{1,n}$. Thus, $\mathrm{span}\{\tilde{e}_{k+1}, \dots, \tilde{e}_n\} = \ker h$, $\ker S = \ker h$. The theorem is proved. ■

The affine manifold (M^n, ∇) is called *complete* if every ∇-geodesic extends infinitely with respect to its affine parameter. A foliation $\mathcal{L}$ on the affine manifold (M^n, ∇) is called *complete* if each leaf $L_x \in \mathcal{L}$ is complete with respect to ∇. The transformation law (9) shows that the connection induced on each leaf does not depend on a choice of transversal distribution Q and, as a consequence, the completeness of $\mathcal{L}$ does not depend on a choice of Q.

P r o o f of the Theorem 2. (1) Suppose $X, Y \in \mathcal{S}$. Then the Codazzi equations (5) imply

$$\nabla_X(S_\alpha Y) - S_\alpha(\nabla_X Y) = \nabla_Y(S_\alpha X) - S_\alpha(\nabla_Y X).$$

Owing to the above, we obtain

$$S_\alpha(\nabla_X Y) - S_\alpha(\nabla_Y X) = 0, \quad S_\alpha(\nabla_X Y - \nabla_Y X) = 0.$$

Thus, $S_\alpha([X, Y]) = 0$ for all $\alpha = \overline{1,k}$ and for all $X \in T_x(M^n)$. Therefore, $[X, Y] \in \mathcal{S}$, and the distribution $\mathcal{S}$ is integrable.

(2) It is enough to prove that $\mathcal{N}$ is totally geodesic, i.e., for any vector fields Y and Z in $\mathcal{N}$, $\nabla_Y Z$ also belongs to $\mathcal{N}$.

Let $Y, Z \in \mathcal{N}$. Then the Codazzi equations (4) imply

$$\begin{aligned}\nabla_X(h^\alpha(Y, Z)) - h^\alpha(\nabla_X Y, Z) - h^\alpha(Y, \nabla_X Z) + \tau^\alpha_\beta(X)h^\beta(Y, Z) = \\ \nabla_Y(h^\alpha(X, Z)) - h^\alpha(\nabla_Y X, Z) - h^\alpha(X, \nabla_Y Z) + \tau^\alpha_\beta(Y)h^\beta(X, Z).\end{aligned}$$

We have $h^\alpha(X, \nabla_Y Z) = 0$ for all $\alpha = \overline{1,k}$ and for all $X \in T_x(M^n)$. Thus, $\nabla_Y Z \in \mathcal{N}$. Hence, $[Y, Z] = \nabla_Y Z - \nabla_Z Y \in \mathcal{N}$, and the distribution $\mathcal{N}$ is integrable and totally geodesic in M^n. Since $Y, Z \in \mathcal{N}$, then $D_Y Z = \nabla_Y Z$, and the leaves are totally geodesic in $\mathbb{R}^{n+k}$.

(3) Let L_x be a leaf of the nullity foliation $\mathcal{FN}$. For $X \in L_x$ we have $S_\alpha X = 0$ as $\ker h \subseteq \ker S$. By (15), we can find a basis of transversal distribution such

that $\tau_\alpha^\beta(X) = 0$. The Weingarten decomposition (2) implies $D_X\xi_\alpha = 0$ for all α and $X \in L_x$.

(4) Let $x_0 \in M^n$ and $X_0 \in \mathcal{N}$. Let $x(t)$ be a geodesic from $x_0 \in L_{x_0}$ in the direction of X_0 and t be its affine parameter. Suppose L_{x_0} is not complete. Then there is $b > 0$ such that $X(b) \notin \mathcal{N}$ and $X(t) \in \mathcal{N}$ for all $t \in [0,\ b)$.

Take an affine (Fermi) geodesic coordinate system in some neighborhood of this geodesic such that the affine tangent frame $U_1, \dots, U_{n-1}, U_n$ of this coordinate system possesses the properties [9]:

$$\begin{gathered} U_n := X = x'(t) \text{ along } x(t), \\ U_{n-\mu+1}(0), \dots, U_n(0) \in \ker h_{x_0}, \\ \nabla_{U_i} U_j \big|_{x(t)} = \Gamma_{ij}^k(x(t))U_k = 0. \end{gathered} \tag{19}$$

Taking into account (19), we have

$$\begin{aligned} X(h^\alpha(U_i,\ U_j)) = (\nabla_X h^\alpha)(U_i,\ U_j) + \\ h^\alpha(\nabla_X U_i,\ U_j) + h^\alpha(U_i,\ \nabla_X U_j) = (\nabla_X h^\alpha)(U_i,\ U_j). \end{aligned} \tag{20}$$

By (15), we can find a transversal frame such that

$$\tau_\beta^\alpha(Y) = 0 \quad \alpha,\ \beta = \overline{1, k},\ Y \in \mathcal{N}. \tag{21}$$

From the Codazzi equations (4), taking into account that $X \in \mathcal{N}$, and relations (19),(21) and (20), we get

$$\begin{aligned} X(h^\alpha(U_i,\ U_j)) = (\nabla_{U_i} h^\alpha)(X,\ U_j) = \\ U_i(h^\alpha(X,\ U_j)) - h^\alpha(\nabla_{U_i} X,\ U_j) - h^\alpha(X,\ \nabla_{U_i} U_j) = 0. \end{aligned} \tag{22}$$

Therefore, $h^\alpha(U_i,\ U_j) = \text{const} = h^\alpha(U_i(0),\ U_j(0))$. Hence, $\dim\ker h_{x(b)} = \dim\ker h_{x(0)}$ and $\{U_{n-\mu+1}(b),\ \dots,\ U_n(b)\}$ is a basis of $\ker h_{x(b)}$. As a consequence, $X(b) \in \mathcal{N}$, which is a contradiction.

The theorem is proved. ■

Corollary 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ $(k < n)$ be an affine immersion with $\dim \operatorname{im} S = k$. Then the submanifold $f(\mathbb{R}^n)$ is foliated by the $(n-k)$-dimensional affine subspaces. The transversal distribution is stationary along the subspaces.*

1. Examples

The following example shows the importance of the maximality of dim im S for the existence of rectilinear generators.

E x a m p l e 1. The submanifold F^3 in $\mathbb{R}^5$, given by

$$\bar{r}(x,y,z)=\begin{pmatrix} x \\ y \\ \cos z \\ \sin z \\ x^2+e^y \end{pmatrix}=\begin{pmatrix} x \\ y \\ 0 \\ 0 \\ x^2+e^y \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ \cos z \\ \sin z \\ 0 \end{pmatrix},$$

is a submanifold with flat connection, non-maximal dimension of the shape operator image, it does not contain rectilinear rulings and is not a graph.

Indeed, put

$$e_1 := \bar{r}'_x = \{1,\ 0,\ 0,\ 0,\ 2x\},$$

$$e_2 := \bar{r}'_y = \{0,\ 1,\ 0,\ 0,\ e^y\},$$

$$e_3 := \bar{r}'_z = \{0,\ 0,\ -\sin z,\ \cos z,\ 0\}.$$

Take a transversal distribution spanned on

$$\xi_1 = \{0,\ 0,\ \cos z,\ \sin z,\ 0\},\ \xi_2 = \{0,\ 0,\ 0,\ 0,\ 1\}.$$

We have

$$\bar{r}''_{xx} = \{0,\ 0,\ 0,\ 0,\ 2\} = 2\xi_2,\ \bar{r}''_{xy} = \bar{0},\ \bar{r}''_{xz} = \bar{0},$$

$$\bar{r}''_{yy} = \{0,\ 0,\ 0,\ 0,\ e^y\} = e^y\xi_2,\ \bar{r}''_{yz} = \bar{0},\ \bar{r}''_{zz} = -\xi_1.$$

As $\nabla_{e_i} e_j = 0$, the connection is flat.

For $X = x^1 e_1 + x^2 e_2 + x^3 e_3$, we have

$$H(X) = \begin{pmatrix} 0 & 0 & -x^3 \\ 2x^1 & e^y x^2 & 0 \end{pmatrix}.$$

In this case the pointwise codimension $q = 2$, the pointwise rank $r = 3$, and $\ker h = 0$.

It is easy to check that $\ker S = \{e_1,\ e_2\}$, $S_1(e_3) = -e_3,\ S_2(e_3) = 0$, i.e. the image of the shape operator is one-dimensional. The submanifold obtained is a product of a graph and a curve.

E x a m p l e 2. A submanifold $f:\mathbb{R}^3\to\mathbb{R}^5$, given by

$$\bar r(x,y,z)=\begin{pmatrix} e^{-x}\cos y - ze^{-x}\sin y\\ e^{-x}\sin y + ze^{-x}\cos y\\ e^{-y}\cos x + ze^{-y}\sin x\\ e^{-y}\sin x - ze^{-y}\cos x\\ z\end{pmatrix},$$

is a ruled one with $\ker S=\ker h$, but is not a cylinder.

We have

$$\bar r'_x=\begin{pmatrix} -e^{-x}\cos y + ze^{-x}\sin y\\ -e^{-x}\sin y - ze^{-x}\cos y\\ -e^{-y}\sin x + ze^{-y}\cos x\\ e^{-y}\cos x + ze^{-y}\sin x\\ 0\end{pmatrix},\ \bar r'_y=\begin{pmatrix} -e^{-x}\sin y - ze^{-x}\cos y\\ e^{-x}\cos y - ze^{-x}\sin y\\ -e^{-y}\cos x - ze^{-y}\sin x\\ -e^{-y}\sin x + ze^{-y}\cos x\\ 0\end{pmatrix}$$

$$\bar r' z=\begin{pmatrix} -e^{-x}\sin y\\ e^{-x}\cos y\\ e^{-y}\sin x\\ -e^{-y}\cos x\\ 1\end{pmatrix}.$$

Take a transversal distribution spanned on

$$\xi_1=\begin{pmatrix} e^{-x}\cos y\\ e^{-x}\sin y\\ -e^{-y}\cos x\\ -e^{-y}\sin x\\ 0\end{pmatrix},\ \xi_2=\begin{pmatrix} e^{-x}\sin y\\ -e^{-x}\cos y\\ e^{-y}\sin x\\ -e^{-y}\cos x\\ 0\end{pmatrix}.$$

Put $e_1=\bar r'_x,\ e_2=\bar r'_y,\ e_3=\bar r'_z$. Then

$$D_{e_1}e_1=\bar r''_{xx}=\begin{pmatrix} e^{-x}\cos y - ze^{-x}\sin y\\ e^{-x}\sin y + ze^{-x}\cos y\\ -e^{-y}\cos x - ze^{-y}\sin x\\ -e^{-y}\sin x + ze^{-y}\cos x\\ 0\end{pmatrix}=\xi_1-z\xi_2.$$

In a similar way,

$$D_{e_1}e_2=\bar r''_{xy}=z\xi_1+\xi_2,\quad D_{e_1}e_3=\bar r''_{xz}=\frac{z\bar r'_x-\bar r'_y}{z^2+1},$$

$$D_{e_2}e_2=\bar r''_{yy}=-\xi_1+z\xi_2,\quad D_{e_2}e_3=\bar r''_{yz}=\frac{\bar r'_x+z\bar r'_y}{z^2+1},\quad D_{e_3}e_3=\bar r''_{zz}=\bar 0.$$

So we have

$$\nabla_{e_i} e_j = 0 \quad i,j = 1,2, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = \frac{z}{z^2+1} e_1 - \frac{1}{z^2+1} e_2,$$
$$\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{z^2+1} e_1 + \frac{z}{z^2+1} e_2.$$

The connection is flat. Indeed,

$$R(e_i, e_j)e_k = 0 \quad (i,j,k = 1,2),$$

$$R(e_1, e_2)e_3 = \nabla_{e_1}\nabla_{e_2} e_3 - \nabla_{e_2}\nabla_{e_1} e_3 - \nabla_{[e_1, e_2]} e_3 =$$
$$\nabla_{e_1}\left(\frac{1}{z^2+1} e_1 + \frac{z}{z^2+1} e_2\right) - \nabla_{e_2}\left(\frac{z}{z^2+1} e_1 - \frac{1}{z^2+1} e_2\right) = 0,$$

$$R(e_1, e_3)e_3 =$$
$$\nabla_{e_1}\nabla_{e_3} e_3 - \nabla_{e_3}\nabla_{e_1} e_3 - \nabla_{[e_1, e_3]} e_3 = -\nabla_{e_3}\left(\frac{z}{z^2+1} e_1 - \frac{1}{z^2+1} e_2\right) =$$
$$-\frac{z^2+1-2z^2}{(z^2+1)^2} e_1 - \frac{z}{z^2+1}\left(\frac{z}{z^2+1} e_1 - \frac{1}{z^2+1} e_2\right) -$$
$$\frac{2z}{(z^2+1)^2} e_2 + \frac{1}{z^2+1}\left(\frac{1}{z^2+1} e_1 + \frac{z}{z^2+1} e_2\right) =$$
$$\frac{(z^2-1-z^2+1)}{(z^2+1)^2} e_1 + \frac{(z-2z+z)}{(z^2+1)^2} e_2 = 0.$$

The components of the affine fundamental form are

$$h^1 = \begin{pmatrix} 1 & z & 0 \\ z & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; h^2 = \begin{pmatrix} -z & 1 & 0 \\ 1 & z & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $X = x^1 e_1 + x^2 e_2 + x^3 e_3$, we have

$$H(X) = \begin{pmatrix} x^1 + zx^2 & zx^1 - x^2 & 0 \\ -zx^1 + x^2 & x^1 + zx^2 & 0 \end{pmatrix}$$

which is equivalent (concerning the rank) to

$$\begin{pmatrix} x^1 + zx^2 & zx^1 - x^2 & 0 \\ (1+z^2)x^2 & (1+z^2)x^1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x^1 & -x^2 & 0 \\ x^2 & x^1 & 0 \end{pmatrix}.$$

Hence, the pointwise codimension $q = \text{rg } H(X) = 2$, the pointwise rank $r = 2$, and $\ker h = \{e_3\}$.

Using the Weingarten decomposition,

$$D_{e_1}\xi_1 = (\xi_1)'_x = \begin{pmatrix} -e^{-x}\cos y \\ -e^{-x}\sin y \\ e^{-y}\sin x \\ -e^{-y}\cos x \\ 0 \end{pmatrix} = -\frac{z}{z^2+1}e_1 + \frac{1}{z^2+1}e_2 - \xi_1 + \xi_2,$$

$$D_{e_2}\xi_1 = (\xi_1)'_y = \begin{pmatrix} -e^{-x}\sin y \\ e^{-x}\cos y \\ e^{-y}\cos x \\ e^{-y}\sin x \\ 0 \end{pmatrix} = -\frac{1}{z^2+1}e_1 - \frac{z}{z^2+1}e_2 - \xi_1 - \xi_2,$$

$$D_{e_1}\xi_2 = D_{e_2}\xi_1,\ D_{e_2}\xi_2 = -D_{e_1}\xi_1,\ D_{e_3}\xi_1 = 0,\ D_{e_3}\xi_2 = 0,$$

we find

$$S_1(e_1) = -S_2(e_2) = \frac{z}{z^2+1}e_1 - \frac{1}{z^2+1}e_2,$$

$$S_1(e_2) = S_2(e_1) = \frac{1}{z^2+1}e_1 + \frac{z}{z^2+1}e_2,\ S_1(e_3) = S_2(e_3) = 0.$$

Therefore, $\ker h = \ker S = \{e_3\}$ and the submanifold is a ruled submanifold, but not a cylinder.

Remark that in contrast to hypersurfaces, in affine case the class of complete, connected regular flat submanifolds is much wider.

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