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Totally geodesic vector fields on pseudo-Riemannian manifolds.

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We consider the submanifolds in the unit tangent bundle of the pseudo-Riemannian manifold generated by the unit vector fields on the base. We have found the second fundamental form of this type of submanifolds with respect to the normal vector field of a special kind. We have derived the equations on totally geodesic non-isotropic unit vector field. We have found all the two-dimensional pseudo-Riemannian manifolds which admit non-isotropic totally geodesic unit vector fields as well as the fields.

Ямпольський О.Л. **Цілком геодезичні векторні поля на псевдо-римановому многовиді.** В роботі розглядаються підмноговиди у дотичному розшаруванні псевдо-риманового многовиду, що породжені векторними полями на базовому многовиді. Знайдено вираз для другої фундаментальної форми такого многовиду відносно поля нормалей спеціальгого виду. Знайдено рівняння на цілком геодезичність неізотропного одиничного векторного поля. Знайдено всі двовимірні многовиди, що дозволяють цілком геодезичні неізотропні векторні поля, а також самі ці поля.

Ямпольский А.Л. **Вполне геодезические векторные поля на псевдо-римановом многообразии.** В работе рассматриваются подмногообразия в касательном расслоении псевдо-риманова многообразия, порождаемые векторными полями на базовом многообразии. Найдено выражение для второй фундаментальной формы этого многообразия относительно поля нормалей специального вида. Получено уравнение на вполне геодезичность неизотропного единичного векторного поля. Найдены все двумерные псевдо-римановы многообразия, допускающие вполне геодезические неизотропные векторные поля, и найдены сами эти векторные поля.

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Introduction

A unit vector field ξ on the Riemannian manifold can be considered as a mapping of the base manifold M into a unit tangent bundle T_1M with the Sasaki metric. Denote by $\xi(M^n)$ the image of the mapping

$$\xi : M^n \to T_1M^n.$$

If the field is smooth, then $\xi(M^n)$ is a smooth submanifold in T_1M^n, at least locally. This fact allows to assign to the vector field the intrinsic properties of $\xi(M^n)$ (e.g. the sectional curvature, the Rici curvature, the scalar curvature) [1], and the extrinsic properties, such as the mean curvature [2, 3, 4] and the totally geodesic property [5]. Thus, the vector field ξ is called *locally-minimal*, if the mean curvature of $\xi(M^n) \subset T_1M^n$ is equal to zero and *totally geodesic*, if this image is the totally geodesic submanifold in the unit tangent bundle.

The Sasaki metric is constructed by using the metric of the base manifold and the Levi-Civita connection. Evidently, the Sasaki metric can be defined on the tangent bundle of the pseudo-Riemannian base manifold as well. If $M^{(p,q)}$ is of signature (p, q), the $TM^{(p,q)}$ is of signature $(2p, 2q)$. The tangent bundle of vectors of fixed length ρ splits into three connected components: $(+)$ the subbundle of the space-like vectors $T_\rho M^{(p,q)}$ $(\rho > 0)$; $(-)$ the subbundle of the time-like vectors $T_\rho M^{(p,q)}$ $(\rho < 0)$; (0) the subbundle of the isotropic vectors $T_0M^{(p,q)}$. The Sasaki metric on $T_0M^{(p,q)}$ is degenerated.

In this paper we consider the non-isotropic vector fields of the length $|\xi|^2 = \pm 1$ on the pseudo-Riemannian manifold $M^{(p,q)}$ and the image $\xi(M^{(p,q)}$ in the corresponding connected component. We derive the system of differential equations which provides the field a totally geodesic property and give a complete solution of this system in case of the two-dimensional pseudo-Riemannian manifold, similar to the description of such fields on the two-dimensional Riemannian manifold [6].

1. Necessary definitions and results

Let M^n be a smooth manifold of dimension n with the local parameters $(u^1, \dots, u^n)$. If we denote by ∂_i $(i = 1, \dots, n)$ the natural tangent frame, then for the any tangent vector ξ we have the decomposition $\xi = \xi^1\,\partial_1 + \dots + \xi^n\,\partial_n$. The parameters $(u^1, \dots, u^n; \xi^1, \dots, \xi^n)$ form the natural local coordinate system on the tangent bundle TM^n.

Let $(M^{(p,q)}, g)$ be a pseudo-Riemannian manifold of the signature (p, q) $(n = p + q)$. Denote by $D\xi^i = d\xi^i + \Gamma^i_{jk}\xi^j du^k$ the covariant differentials of tangent vector ξ coordinates with respect to the Levi-Civita connection Γ^i_{jk} of the metric g. Similar to the Riemannian case, the Sasaki metric on $TM^{(p,q)}$ is given by

$$d\sigma^2 = g_{ik}du^i du^k + g_{ik}D\xi^i D\xi^k$$

or, in invariant form,

$$d\sigma^2 = ds^2 + |D\xi|^2_g.$$

The signature of $d\sigma^2$ is equal to $(2p, 2q)$.

At each point $(Q,\xi) \in TM^{(p,q)}$ we have the following decomposition

$$T_{(Q,\xi)}(TM) = \mathcal{V}_{(Q,\xi)} \oplus \mathcal{H}_{(Q,\xi)},$$

where the subspace $\mathcal{V}_{(Q,\xi)}$ is tangent to the fiber $T_Q M^{(p,q)}$ and the subspace $\mathcal{H}_{(Q,\xi)}$ is transversal to the fiber $T_Q M^{(p,q)}$. The subspaces $\mathcal{V}_{(Q,\xi)}$ and $\mathcal{H}_{(Q,\xi)}$ are called *vertical* and *horizontal* subspaces respectively. Denote by π_* and K the differential of the projection $\pi : TM \to M$ and the connection map [7] respectively. Then

$$\mathcal{V}_{(Q,\xi)}(TM) = \ker \pi_*, \qquad \mathcal{H}_{(Q,\xi)}(TM) = \ker K.$$

If $X = X^i \partial_i$ is a vector field on the base, then

$$X^h = X^i\,\partial_i - \Gamma^i_{jk}\xi^j X^k\,\partial_{n+i}, \qquad X^v = X^i\,\partial_{n+i}$$

and are vector fields on $TM^{(p,q)}$. They are called by *horizontal* and *vertical* lifts of X respectively.

The scalar product with respect to the Sasaki metric is given by

$$\big\langle\, \tilde X, \tilde Y \,\big\rangle_S = \big\langle\, \pi_*\tilde X, \pi_*\tilde Y \,\big\rangle_g + \big\langle\, K\tilde X, K\tilde Y \,\big\rangle_g.$$

The following Lemma was proved by A. Gray [9] (the Riemannian case was treated by O. Kowalski [8]).

Lemma 1 *Let $(M^{(p,q)}, g)$ be a pseudo-Riemannian manifold, ∇ is the Levi-Civita connection of M, $R(X,Y)Z$ is a curvature tensor of g. Then the Levi-Civita connection $\tilde\nabla$ of the Sasaki metric on $TM^{(p,q)}$ is completely defined by*

$$\tilde\nabla_{X^h} Y^h = \big(\nabla_X Y\big)^h - \tfrac12\big(R(X,Y)\xi\big)^v, \quad \tilde\nabla_{X^h} Y^v = (\nabla_X Y)^v + \tfrac12\big(R(\xi,Y)X\big)^h,$$

$$\tilde\nabla_{X^v} Y^h = \tfrac12\big(R(\xi,X)Y\big)^h, \qquad \tilde\nabla_{X^v} Y^v = 0.$$

at each point $(Q,\xi) \in TM$.

Consider a subbundle in $TM^{(p,q)}$, formed by the vectors of the length $|\xi|^2 = \epsilon = \pm 1$. These subbundles are hypersurfaces in $TM^{(p,q)}$ with the pull-back metrics and are called the bundles of unit space-like vectors for $\epsilon = +1$ and the bundle unit time-like vectors for $\epsilon = -1$, respectively. In what follows, we refer to these subbundles as *the bundle of ϵ-unit vectors* and denote by $T_\epsilon M$.

2. The second fundamental form of a unit vector field.

Let ξ be a vector field on $M^{(p,q)}$. The vector field ξ is called *a unit space-like*, if $|\xi|^2 = +1$ and *a unit time-like*, if $|\xi|^2 = -1$, respectively. We call such a field by *ϵ-unit vector field* for brevity.

Consider the ϵ-unit vector field ξ as a (local) mapping

$$\xi : M^{(p,q)} \to T_\epsilon M^{(p,q)}. \tag{1}$$

The image $\xi(M^{(p,q)})$ is a submanifold in ϵ-unit tangent bundle $T_\epsilon M^{(p,q)}$ given locally by

$$u^i = u^i; \quad \xi^i = \xi^i(u_1, \dots, u^n) \tag{2}$$

with respect to natural coordinates in $TM^{(p,q)}$. The Sasaki metric of the tangent bundle $TM^{(p,q)}$ defines a metric on $T_\epsilon M^{(p,q)}$ and on $\xi(M^{(p,q)})$ in a chain of inclusions

$$\xi(M^{(p,q)}) \subset T_\epsilon M^{(p,q)} \subset TM^{(p,q)}.$$

For any vector field X on M, the differential of (1) acts by

$$\xi_* X = X^h + (\nabla_X \xi)^v = X^h - (A_\xi X)^v,$$

where $A_\xi X = -\nabla_X \xi$ is the non-holonomic Waingarten operator.

In these terms, the metric on $\xi(M)$ takes the form

$$\big\langle\, \xi_* X, \xi_* Y \,\big\rangle_S = \big\langle\, X, Y \,\big\rangle_g + \big\langle\, A_\xi X, A_\xi Y \,\big\rangle_g. \tag{3}$$

The *conjugate Weingarten operator* A^t_ξ is defined by

$$\big\langle\, A_\xi X, Y \,\big\rangle_g = \big\langle\, X, A^t_\xi Y \,\big\rangle_g.$$

In local coordinates, $\big(A^t_\xi\big)^k_i = -g^{is}\nabla_s \xi^j g_{jk}$ and hence it is well-defined since the metric g is non-degenarate.

It is easy to check that the *normal bundle* of $\xi(M^{(p,q)}) \subset T_\epsilon M^{(p,q)}$ is spanned on the vector fields of the form

$$\tilde N = (A^t_\xi N)^h + (N)^v,$$

where $N \in \xi^\perp$.

Now we can formulate our basic lemma.

Lemma 2 *With respect to the normal vector field* $\tilde N = (A^t_\xi N)^h + (N)^v$ *on* $\xi(M^{(p,q)}) \subset T_\epsilon M^{(p,q)}$*, the second fundamental form of* $\xi(M^{(p,q)})$ *is given by*

$$\tilde\Omega_{\tilde N}(\xi_* X, \xi_* Y) = \\ -\frac{1}{2}\big\langle\, (\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X + A_\xi\big(R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X\big), N \,\big\rangle \tag{4}$$

where $(\nabla_X A_\xi)Y = \nabla_{\nabla_X Y}\xi - \nabla_X \nabla_Y \xi$ *and* $N \in \xi^\perp$.

The prof is straightforward and repeats the proof in the Riemannian case [10].

As a consequence, we receive the following condition on $\xi(M^{(p,q)}) \subset T_\epsilon M^{(p,q)}$ to be totally geodesic.

Lemma 3 *The submanifold $\xi(M^{(p,q)}) \subset T_\epsilon M^{(p,q)}$ is totally geodesic if and only if the field ξ satisfies*

$$(\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X + A_\xi\big(R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X\big) - 2\,\epsilon\,\langle\, A_\xi X, A_\xi Y\,\rangle\xi = 0 \quad (5)$$

for any vector fields X, Y on $M^{(p,q)}$.

Indeed, the vertical lift ξ^v is the field of unit normals on $T_\epsilon M$. That is why $\tilde\Omega \equiv 0$ if and only if

$$(\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X + A_\xi\big(R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X\big) = \lambda\,\xi.$$

One can find the function λ by multiplying the latter equation by ξ and noticing that

$$\big\langle\, (\nabla_X A_\xi)Y, \xi\,\big\rangle = \big\langle\, (\nabla_Y A_\xi)X, \xi\,\big\rangle = \big\langle\, A_\xi X, A_\xi Y\,\big\rangle, \qquad |\xi|^2 = \epsilon.$$

3. Totally geodesic unit vector fields on $M^{(1,1)}$.

The equation (5) always admits a trivial solution $\nabla\xi = 0$. In the two-dimensional case it means that the manifold is flat. Remark also, that if ξ is the totally geodesic field, then $-\xi$ is the totally geodesic as well. Similar to the Riemannian case [6], we can solve the equation (5) completely with respect to the field and the base manifold in the following terms.

Theorem 1 *Let $M^{(1,1)}$ be a pseudo-Riemannian non-flat 2-manifold. Suppose $M^{(1,1)}$ admits the totally geodesic ϵ-unit vector field ξ. Then either*

- *$M^{(1,1)}$ admits a metric of the form*

  $$ds^2 = \epsilon(du^2 - sin^2\alpha\, dv^2),$$

  where $\alpha = \alpha(u)$ is a solution of the differential equation $\dfrac{d\alpha}{du} = -1 - \dfrac{a-1}{\cos\alpha}$. The corresponding totally geodesic ϵ-unit vector field is

  $$\pm\xi = \cosh(av + \omega_0)\,\partial_u + \frac{\sinh(av + \omega_0)}{\sin\alpha(u)}\,\partial_v.$$

  or

- *$M^{(1,1)}$ admits a metric of the form*

  $$ds^2 = \epsilon(du^2 - \sinh^2\alpha\, dv^2),$$

  where $\alpha = \alpha(u)$ is a solution of the differential equation $\dfrac{d\alpha}{du} = 1 - \dfrac{a+1}{\cosh\alpha}$. The corresponding totally geodesic ϵ-unit vector field is

  $$\pm\xi = \mathrm{sh}(av + \omega_0)\,\partial_u + \frac{\mathrm{ch}(av + \omega_0)}{\sinh\alpha(u)}\,\partial_v.$$

Proof. Let e_1, e_2 be an orthonormal frame on $M^{(1,1)}$. Set

$$|e_1|^2 = \epsilon, \qquad |e_2|^2 = -\epsilon, \qquad \epsilon = \pm 1.$$

Let ξ be the ϵ-unit vector field on $M^{(1,1)}$. Denote by η a $(-\epsilon)$-unit vector field which is orthogonal to ξ. Then we have

$$A_\xi e_1 = \lambda_1 \eta, \quad A_\xi e_2 = \lambda_2 \eta.$$

If $M^{(1,1)}$ is non-flat, then there exists a non-isotropic orthonormal frame e_1, e_2 such that

$$A_\xi e_1 = 0, \quad A_\xi e_2 = \lambda \eta, \tag{6}$$

Indeed, we take

$$X_0 = \lambda_2\, e_1 - \lambda_1\, e_2, \quad Y_0 = \lambda_1\, e_1 - \lambda_2\, e_2.$$

Then

$$|X_0|^2 = \epsilon(\lambda_2^2 - \lambda_1^2), \quad |Y_0|^2 = -\epsilon(\lambda_2^2 - \lambda_1^2) = -|X_0|^2, \quad \big\langle X_0, Y_0 \big\rangle = 0$$

and

$$A_\xi X_0 = 0, \quad A_\xi Y_0 = \epsilon |X_0|^2\, \eta.$$

If X_0 is isotropic, than the field ξ is parallel on $M^{(1,1)}$ and hence $M^{(1,1)}$ is flat. By normalizing (X_0, Y_0), we get the required frame.

Take a frame e_1, e_2 which satisfies (6). Put

$$\nabla_{e_1} e_1 = k_1\, e_2, \quad \nabla_{e_2} e_2 = k_2\, e_1.$$

Then

$$\nabla_{e_1} e_2 = k_1 e_1, \quad \nabla_{e_2} e_1 = k_2\, e_2.$$

As $\nabla_{e_1}\xi = 0$ and $\nabla_{e_2}\xi = -A_\xi e_2 = -\lambda\eta$, we have

$$\nabla_{e_1}\eta = 0, \quad \nabla_{e_2}\eta = -\lambda\, \xi.$$

As $(\nabla_{e_i} A_\xi) e_k = \nabla_{e_i}(A_\xi e_k) - A_\xi(\nabla_{e_i} e_k)$, we have

$$(\nabla_{e_1} A_\xi) e_1 = \nabla_{e_1}(A_\xi e_1) - A_\xi(\nabla_{e_1} e_1) = -k_1 A_\xi e_2 = -k_1 \lambda$$

$$(\nabla_{e_1} A_\xi) e_2 = \nabla_{e_1}(A_\xi e_2) - A_\xi(\nabla_{e_1} e_2) = \nabla_{e_1}(\lambda\eta) - A_\xi(k_1 e_1) = e_1(\lambda)\eta.$$

$$(\nabla_{e_2} A_\xi) e_1 = \nabla_{e_2}(A_\xi e_1) - A_\xi(\nabla_{e_2} e_1) = -A_\xi(k_2 e_2) = -k_2 \lambda$$

$$(\nabla_{e_2} A_\xi) e_2 = \nabla_{e_2}(A_\xi e_2) - A_\xi(\nabla_{e_2} e_2) = \nabla_{e_2}(\lambda\eta) = e_2(\lambda)\eta - \lambda^2 \xi.$$

As $R(e_1, e_2)\xi = (\nabla_{e_2} A_\xi) e_1 - (\nabla_{e_1} A_\xi) e_2$, we find

$$R(e_1, e_2)\xi = -\big(k_2\lambda + e_1(\lambda)\big)\eta = -K\eta,$$

where we have put $K = k_2\lambda + e_1(\lambda)$. Hence,

$$\left\langle R(e_1, e_2)\xi, \eta \right\rangle = -K|\eta|^2.$$

In dependence on the orientation of the pair (ξ, η), we have

$$\left\langle R(e_1, e_2)\xi, \eta \right\rangle = -K|\eta|^2 = \epsilon K \qquad \text{for } (+) \text{ orientation,}$$

$$\left\langle R(e_1, e_2)\xi, \eta \right\rangle = -K|\eta|^2 = -\epsilon K \quad \text{for } (-) \text{ orientation.}$$

Uniformly,

$$\left\langle R(e_1, e_2)\xi, \eta \right\rangle = (-1)^{s+1}\epsilon K, \tag{7}$$

where $s = 0$ for $(+)$ orientation and $s = 1$ for $(-)$ orientation of (ξ, η) frame.

As a consequence,

$$R(\xi, A_\xi e_1)e_1 = 0, \quad R(\xi, A_\xi e_1)e_2 = 0,$$

$$R(\xi, A_\xi e_2)e_1 = -\epsilon\lambda\left\langle R(e_1, e_2)\xi, \eta \right\rangle e_2 = (-1)^s \lambda K e_2,$$

$$R(\xi, A_\xi e_2)e_2 = \epsilon\lambda\left\langle R(e_2, e_1)\xi, \eta \right\rangle e_1 = (-1)^s \lambda K e_1,$$

$$A_\xi R(\xi, A_\xi e_1)e_1 = 0, \qquad A_\xi R(\xi, A_\xi e_1)e_2 = 0,$$

$$A_\xi R(\xi, A_\xi e_2)e_1 = (-1)^s \lambda^2 K\eta, \quad A_\xi R(\xi, A_\xi e_2)e_2 = 0.$$

If we put sequentially $(X = e_1, Y = e_1)$, $(X = e_1, Y = e_2)$ and $(X = e_2, Y = e_2)$ and plug the results above into (5), then we get respectively

$$\begin{aligned} &-k_1\lambda = 0, \\ &(e_1(\lambda) - k_2\lambda) + (-1)^s \lambda^2 K = 0, \\ &e_2(\lambda) = 0. \end{aligned} \tag{8}$$

By hypothesis, the manifold is not flat and so, ξ is not parallel. Therefore, $\lambda \neq 0$ and from $(8)_1$ we conclude $k_1 = 0$. We can rewrite the equation $(8)_2$ as follows

$$e_1(\lambda) - k_2\lambda + (-1)^s \lambda^2 (k_2\lambda + e_1(\lambda)) = 0$$

or

$$k_2 = \frac{1 + (-1)^s \lambda^2}{1(-1)^s \lambda^2} \, \frac{e_1(\lambda)}{\lambda}.$$

Thus, we reduce the system (8) to

$$\begin{cases} k_1 = 0, \\ k_2 = \dfrac{1 + (-1)^s \lambda^2}{1(-1)^s \lambda^2} \, \dfrac{e_1(\lambda)}{\lambda}, \\ e_2(\lambda) = 0. \end{cases} \tag{9}$$

As $k_1 = 0$, the integral trajectories of e_1 are ϵ-geodesics. Therefore, it is possible to introduce the local semi-geodesic system of coordinate on $M^{(1,1)}$ and simplify the metric to the form

$$ds^2 = \epsilon(du^2 - f^2(u,v)\,dv^2)$$

with $e_1 = \partial_1,\ e_2 = \frac{1}{f}\,\partial_2$. After this,

$$k_2 = \frac{f_u}{f}$$

In case $s = 1$, we have

$$\xi = \operatorname{ch}\omega\, e_1 + \operatorname{sh}\omega\, e_2, \qquad |\xi|^2 = \epsilon,$$
$$\eta = \operatorname{sh}\omega\, e_1 + \operatorname{ch}\omega\, e_2$$

and the equation $(9)_2$ yields

$$\frac{f_u}{f} = \frac{1-\lambda^2}{1+\lambda^2}\frac{e_1(\lambda)}{\lambda}. \tag{10}$$

Put $\lambda = \tan(\alpha/2)$, where $\alpha = \alpha(u,v)$ is some function. From $(9)_3$ it follows that $\partial_v\alpha = 0$ and the equation (10) takes the form

$$\frac{\partial_u f}{f} = \operatorname{ctg}\alpha\ \partial_u\alpha.$$

The solution is $f = C(v)\sin\alpha$, where $C(v)$ is arbitrary positive function. By making the parameter change, we reduce the metric to the form

$$ds^2 = \epsilon(du^2 - \sin^2\alpha\, dv^2),$$

where $\alpha = \alpha(u)$. Moreover, we have

$$\nabla_{e_2}\xi = (e_2(\omega) + k_2)\eta = \left(\frac{\partial_v\omega}{\sin\alpha} + \operatorname{ctg}\alpha\ \partial_u\alpha\right)\eta.$$

$$\operatorname{tg}(\alpha/2) = \lambda = \frac{\partial_v\omega}{\sin\alpha} + \operatorname{ctg}\alpha\ \partial_u\alpha$$

and hence

$$\partial_v\omega = \sin\alpha(\operatorname{tg}(\alpha/2) - \operatorname{ctg}\alpha\ \partial_u\alpha).$$

As α does not depend on v, we conclude $\omega = av + \omega_0$. Thus, the function α satisfies

$$\cos\alpha\ \partial_u\alpha - 2\sin^2(\alpha/2) = -a$$

or, equivalently,

$$\frac{d\alpha}{du} = -1 - \frac{a-1}{\cos\alpha}.$$

As a conclusion,

$$\xi = \operatorname{ch}(av+\omega_0)\,\partial_u + \frac{\operatorname{sh}(av+\omega_0)}{\sin\alpha(u)}\,\partial_v.$$

In case $s = 2$, we have

$$\begin{aligned} \xi &= \operatorname{sh}\omega\, e_1 + \operatorname{ch}\omega\, e_2, \qquad |\xi^2| = -\epsilon, \\ \eta &= \operatorname{ch}\omega\, e_1 + \operatorname{sh}\omega\, e_2 \end{aligned}$$

and the equation $(9)_2$ yields

$$\frac{f_u}{f} = \frac{1+\lambda^2}{1-\lambda^2}\,\frac{e_1(\lambda)}{\lambda}. \tag{11}$$

Put

$$\lambda = \operatorname{th}(\alpha/2),$$

where $\alpha = \alpha(u,v)$ is some function. From $(8)_3$ it follows that

$$\partial_v\alpha = 0$$

and (11) yields

$$\frac{\partial_u f}{f} = \operatorname{cth}\alpha\ \partial_u\alpha.$$

The solution is

$$f = C(v)\operatorname{sh}\alpha,$$

where $C(v)$ is arbitrary positive function. By making a parameter change, we reduce the metric to the form

$$ds^2 = \epsilon(du^2 - \operatorname{sh}^2\alpha\, dv^2),$$

where $\alpha = \alpha(u)$. Moreover, we have

$$\nabla_{e_2}\xi = \nabla_{e_2}(\operatorname{sh}\omega e_1 + \operatorname{ch}\omega e_2) = (e_2(\omega)+k_2)\eta = \left(\frac{\partial_v\omega}{\operatorname{sh}\alpha} + \operatorname{cth}\alpha\ \partial_u\alpha\right)\eta.$$

This means that

$$\tanh(\alpha/2) = \lambda = \frac{\partial_v\omega}{\operatorname{sh}\alpha} + \operatorname{cth}\alpha\ \partial_u\alpha$$

or, equivalently,

$$\partial_v\omega = \operatorname{sh}\alpha(\tanh(\alpha/2) - \operatorname{cth}\alpha\,\partial_u\alpha).$$

As α does not depend on v,

$$\omega = av + \omega_0.$$

Hence, α satisfies

$$\operatorname{ch}\alpha\ \partial_u\alpha - 2\operatorname{sh}^2(\alpha/2) = -a$$

or, after the transformations,

$$\frac{d\alpha}{du} = 1 - \frac{a+1}{\operatorname{ch}\alpha}.$$

As a conclusion,

$$\xi = \operatorname{sh}(av + \omega_0)\,\partial_u + \frac{\operatorname{ch}(av+\omega_0)}{\operatorname{sh}\alpha(u)}\,\partial_v.$$

The Theorem is proved.

■

If K is the Gaussian curvature of $M^{(1,1)}$, then $K = -\epsilon\,\alpha_u$. Indeed, for the metric $ds^2 = \epsilon(du^2 - f^2\,dv^2)$ we have

$$K = -\epsilon\,\frac{f_{uu}}{f}.$$

Therefore, in the case

$$ds^2 = \epsilon(du^2 - \sin^2\alpha\,dv^2), \qquad \frac{d\alpha}{du} = -1 - \frac{a-1}{\cos\alpha}$$

we have

$$\begin{aligned} f_u &= \cos\alpha\,\alpha_u = \cos\alpha(-1 - \tfrac{a-1}{\cos\alpha}) = -\cos\alpha - (a-1),\\ f_{uu} &= \sin(\alpha)\,\alpha_u = f\,\alpha_u \end{aligned}$$

and hence

$$K = -\epsilon\,\alpha_u = \epsilon\left(1 + \frac{a-1}{\cos\alpha}\right).$$

In the case

$$ds^2 = \epsilon(du^2 - \operatorname{sh}^2\alpha\,dv^2), \qquad \frac{d\alpha}{du} = 1 - \frac{a+1}{\operatorname{ch}\alpha},$$

the similar calculations yield

$$K = -\epsilon\,\alpha_u = -\epsilon\left(1 - \frac{a+1}{\operatorname{ch}\alpha}\right).$$

In the case of the *Riemannian* 2- manifold of constant curvature, the totally geodesic unit vector field exists on the manifold of the Gaussian curvature $K = 1$ [1]. In contrast, in the case of the *pseudo-Riemannian* 2-manifold, the totally geodesic ϵ-unit vector field exists both for $K = 1$ and for $K = -1$ (for corresponding values of a). The corresponding totally geodesic submanifolds $\xi(M^{(1,1)})$ are located in $T_{+1}M$ or $T_{-1}M$, respectively.

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