Totally geodesic property of the Hopf vector field.*

Yampolsky A.

Abstract

We prove that the Hopf vector field is a unique one among geodesic unit vector fields on spheres such that the submanifold generated by the field is totally geodesic in the unit tangent bundle with Sasaki metric. As application, we give a new proof of stability (instability) of the Hopf vector field with respect to volume variation using standard approach from the theory of submanifolds and find exact boundaries for the sectional curvature of the Hopf vector field. [1]

Introduction

Let (M, g) be $(n+1)$-dimensional Riemannian manifold with metric g and ξ a fixed unit vector field on M. Consider ξ as a local mapping $\xi : M \to T_1M$. Then the image $\xi(M)$ is a submanifold in the unit tangent sphere bundle T_1M. The Sasaki metric on the tangent bundle TM induces the Riemannian metric on T_1M and on $\xi(M)$ as well. So, one may use the geometrical properties of this submanifold to determine geometrical characteristics of a unit vector field. Namely, a unit vector field ξ is said to be *minimal* if $\xi(M)$ is of minimal volume with respect to the induced metric [6, 4]; ξ is *totally geodesic* if $\xi(M)$ is totally geodesic submanifold in T_1M. A complete description of totally geodesic vector fields on 2-dimensional manifolds of constant curvature has been given in [14]. It was proved that only unit sphere (among non-flat space forms) admits totally geodesic unit vector fields (which are *non-geodesic* ones). In contrast to this case, on a 3-dimensional sphere the Hopf (and therefore, geodesic) unit vector field is a unique one with globally minimal volume [6]. Since totally geodesic submanifold is always minimal, it was natural to suspict that the Hopf vector field is totally geodesic in fact. In addition, for the spheres of greater dimensions the Hopf vector fields stays minimal but become unstable [10, 6]. Nevertheless, something should distinguish the Hopf vector fields. And this *something*, as we prove here, is their totally geodesic property. Namely, (Theorem 2.1)

Let ξ be a unit geodesic vector field on a unit sphere S^{n+1}. Then ξ is totally geodesic vector field if and only if $n = 2m$ and ξ is a Hopf vector field.[2]

Using this result we give another proof of stability or instability of the Hopf vector fields with respect to volume functional (cf. [5])using a standard formula

*Acta Math. Hungar. 101, 1-2 (2003), 73-92.

[1] *Keywords and phrases:* Sasaki metric, vector field, totally geodesic submanifolds.
AMS subject class: Primary 54C40,14E20; Secondary 46E25, 20C20

[2] The result is true with one additional condition (see below)

for the second volume variation from the theory of the submanifolds (Theorems 3.1 and 3.2).

The Hopf vector field on S^{2m+1} is **stable** *for $m = 1$ and* **unstable** *for $m > 1$.*

We also find an exact boundaries for the sectional cutvature of this field (Theorem 4.1).

Sectional curvature of $\xi(S^{2m+1})$ varies between $\frac{1}{4}$ and $\frac{5}{4}$. Minimal bound the curvature achieves on ξ-tangential lift (3) of ξ-section and maximal on ξ-tangential lift of φ-section in terms of contact metric geometry.

1 Some preliminaries

Let (M, g) be $(n+1)$-dimensional Riemannian manifold with metric g. Denote by $\langle \cdot, \cdot \rangle$ a scalar product with respect to g and by ∇ the Levi-Civita connection on M.

The *Sasaki metric* on TM is defined by the following scalar product: if $\tilde{X}, \tilde{Y} \in TTM$, then

$$\langle\langle \tilde{X}, \tilde{Y} \rangle\rangle = \langle \pi_* \tilde{X}, \pi_* \tilde{Y} \rangle + \langle K\tilde{X}, K\tilde{Y} \rangle \tag{1}$$

where $\pi_* : TTM \to TM$ is a differential of projection $\pi : TM \to M$ and $K : TTM \to TM$ is the *connection map.*

Let ξ be a unit vector field on M. A vector field $\tilde{X} \in TTM$ is tangent to $\xi(M)$ if and only if [12]

$$\tilde{X} = (\pi_* \tilde{X})^h + (\nabla_{\pi_* \tilde{X}} \xi)^v,$$

where $(\cdot)^h$ and $(\cdot)^v$ mean *horizontal* and *vertical* lifts of fields into the tangent bundle.

It is well known that ξ^v is a unit normal vector field on T_1M. Let X be tangent to M, then

$$X^t = X^v - \langle X, \xi \rangle \xi^v,$$

is always tangent to T_1M and is called the *tangential lift* of X [1]. It is easy to see that

$$\langle\langle X^t, Y^t \rangle\rangle = \langle X, Y \rangle - \langle X, \xi \rangle \langle Y, \xi \rangle.$$

Introduce, now, a notion of ξ*-tangential* and ξ*-normal* lifts with respect to given field ξ. We proceed in the following way.

Introduce a *shape operator* A_ξ for the field ξ by

$$A_\xi X = -\nabla_X \xi,$$

where X is arbitrary vector field on M.

Define a *conjugate shape operator* A^*_ξ by

$$\langle A^*_\xi X, Y \rangle = \langle X, A_\xi Y \rangle. \tag{2}$$

Let X, Y be the vector fields on M. Define ξ-tangential lift X^τ_ξ and ξ-normal lift Y^ν_ξ of X and Y respectively by

$$X^\tau_\xi = X^h - (A_\xi X)^t, \quad Y^\nu_\xi = (A^*_\xi Y)^h + Y^t. \tag{3}$$

Then, X_ξ^τ is evidently tangent to $\xi(M)$. Moreover,

$$\langle\langle X_\xi^\tau, Y_\xi^\nu\rangle\rangle = \langle\langle X^h, (A_\xi^* Y)^h\rangle\rangle - \langle\langle (A_\xi X)^t, Y^t\rangle\rangle =$$
$$\langle X, A_\xi^* Y\rangle - \langle A_\xi X, Y\rangle + \langle A_\xi X, \xi\rangle\langle Y, \xi\rangle = \langle A_\xi X, Y\rangle - \langle A_\xi X, Y\rangle \equiv 0$$

and therefore the linear space of all ξ-normal lifts coincides with the normal space of $\xi(M)$ at each point.

To construct a natural tangent and normal orthonormal frames for $\xi(M)$, one can use a singular decomposition of the shape operator A_ξ, based on the following linear algebra result ([9], Theorem 7.3.5 and Exercise 7.3.5) which we present here in a slightly modified form.

Theorem 1.1 *A matrix $A \in M_{m,n}$ of rank k may be represented in the form*

$$A = F\Sigma E^*,$$

where $F \in M_m$ and $E \in M_n$ are unitary matrices.

The matrix $\Sigma = [\sigma_{ij}] \in M_{m,n}$ is such that $\sigma_{ij} = 0$, $i \neq j$, $\sigma_{11} \geq \sigma_{22} \geq \dots \geq \sigma_{kk} > \sigma_{k+1,k+1} = \dots = \sigma_{qq} = 0$, $q = \min\{m, n\}$. The values $\{\sigma_{ii}\} \equiv \{\lambda_i\}$ are non-negative square roots of eigenvalues of the matrix AA^* and hence are uniquely defined. The columns of the matrix F are the eigenvectors of the matrix AA^* and the columns of the matrix E are the eigenvalues of the matrix A^*A. Moreover, $A^* f_i = \lambda_i e_i$ and $Ae_i = \lambda_i f_i$ for $i = 1, \dots, k$. If the matrix A is real, then F, Σ and E can be real.

The columns of the matrices F and E are called respectively *left* and *right singular vectors* of matrix A. The values λ_i are called *singular values* of the matrix A.

Set $A = A_\xi$ and apply Theorem 1.1. Since $A_\xi^*\xi = 0$ for any unit vector field ξ , there exist an orthonormal local frames $e_0, e_1, \dots, e_n$ and $f_0 = \xi$, $f_1, \dots, f_n$ on M such that

$$A_\xi\, e_0 = 0,\ A_\xi\, e_\alpha = \lambda_\alpha f_\alpha, \quad A_\xi^*\, f_0 = 0,\ A_\xi^*\, f_\alpha = \lambda_\alpha e_\alpha,\ \alpha = 1, \dots, n,$$

where $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq 0$ are the real-valued functions.

It is natural to call the functions λ_i $(i = 1, \dots, n)$ the *singular principal curvatures* of the field ξ with respect to chosen singular frame. Remark, that if necessary one may use the signed singular values fixing the directions of the vectors of the singular frame. Setting $\lambda_0 = 0$, we may rewrite the relations on singular frames in a unified form

$$\begin{aligned} &A_\xi\, e_i = \lambda_i f_i, \quad A_\xi^*\, f_i = \lambda_i e_i, \qquad i = 0, 1, \dots, n, \\ &\lambda_0 = 0,\ \lambda_1, \dots, \lambda_n \geq 0. \end{aligned} \tag{4}$$

The following lemma is easy to prove using (2) and (4).

Lemma 1.1 *[12] At each point of $\xi(M) \subset TM$ the orthonormal frames*

$$\begin{aligned} &\tilde e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(e_i^h - \lambda_i f_i^v), \qquad i = 0, 1, \dots, n, \\ &\tilde n_{\sigma|} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big(\lambda_\sigma e_\sigma^h + f_\sigma^v\ \big), \quad \sigma = 1, \dots, n \end{aligned} \tag{5}$$

form the orthonormal frames in the tangent space of $\xi(M)$ and in the normal space of $\xi(M)$ respectively.

Set
$$(\nabla_X A_\xi)\, Y = \nabla_X (A_\xi Y) - A_\xi \nabla_X Y.$$
If we introduce a *half tensor* of Riemannian curvature as
$$r(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi, \tag{6}$$
we can easily see that
$$-(\nabla_X A_\xi)\, Y = r(X,Y)\xi. \tag{7}$$

Now, we are able to formulate a basic lemma for our considerations.

Lemma 1.2 *[12] The components of second fundamental form of $\xi(M) \subset T_1 M$ with respect to the frame (5) are given by*

$$\begin{aligned}\tilde\Omega_{\sigma|ij} = \quad & \tfrac{1}{2}\Lambda_{\sigma ij}\Big\{\big\langle r(e_i,e_j)\xi + r(e_j,e_i)\xi, f_\sigma\big\rangle + \\ & \lambda_\sigma \left[\lambda_j \big\langle R(e_\sigma,e_i)\xi, f_j\big\rangle + \lambda_i \big\langle R(e_\sigma,e_j)\xi, f_i\big\rangle\right]\Big\},\end{aligned}$$

where $\Lambda_{\sigma ij} = [(1+\lambda_\sigma^2)(1+\lambda_i^2)(1+\lambda_j^2)]^{-1/2}$ $(i,j = 0,1,\dots,n;\ \sigma = 1,\dots,n)$.

Remark. We say that the given unit vector field is *holonomic* if ξ is a field of unit normals for a family of hypersurfaces in M and *non-holonomic* otherwise. If the integral trajectories of ξ are geodesics in M then ξ is called *geodesic* vector field. Evidently, in the case of holonomic geodesic unit vector field, A_ξ becomes a usual shape operator for each hypersurface. In this case A_ξ is self-adjoint (symmetric), i.e.
$$\big\langle A_\xi X, Y\big\rangle = \big\langle X, A_\xi Y\big\rangle \text{ and thus } A_\xi^* = A_\xi$$
with respect to some orthonormal frame. Let $R(X,Y)\,\xi = \big[\nabla_X, \nabla_Y\big]\xi - \nabla_{[X,Y]}\,\xi$ be a curvature tensor of M. The non-holonomic shape operator satisfies the *non-holonomic Codazzi* equation
$$R(X,Y)\xi = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y = r(X,Y)\xi - r(Y,X)\xi. \tag{8}$$
These facts justify the terminology for the operator A_ξ and the tensor r.

2 Proof of the main result.

In this section we will prove of the following theorem.

Theorem 2.1 *Let ξ be a unit geodesic vector field on a unit sphere S^{n+1}. Then ξ is totally geodesic vector field if and only if $n = 2m$ and ξ is a Hopf vector field*[3].

[3]The nesessary part of the result is true if the vector field satisfies a condition of covarian normality, that is $A_\xi A_\xi^* = A_\xi^* A_\xi$.

Sufficient part of the proof.

The sufficiency is a consequence of more general considerations involving the properties of Killing and Sasakian structure vector fields. Remind that the Hopf vector field is both Killing and Sasakian structure characteristic vector field.

We begin with Killing vector fields. Let M be $(n+1)$-dimensional a Riemannian manifold admitting a Killing vector field ξ. Then A_ξ is skew-symmetric

$$\big\langle A_\xi X, Y\big\rangle = -\big\langle X, A_\xi Y\big\rangle \text{ and thus } A_\xi^* = -A_\xi \tag{9}$$

with respect to some orthonormal frame.

The field ξ is necessarily *geodesic* and evidently $A_\xi\xi = 0$. For the singular frames we have $e_0 = f_0 = \xi$ and $e_\alpha, f_\alpha \in \xi^\perp$ for all $\alpha = 1, \dots, n$. Moreover, set $2m = \operatorname{rank} A_\xi$. There exists an orthonormal frame

$$e_0 = \xi,\ e_1, \dots, e_m,\ e_{m+1}, \dots, e_{2m},\ e_{2m+1}, \dots, n$$

such that

$$\begin{aligned} &A_\xi\, e_\alpha = \lambda_\alpha e_{m+\alpha}, \quad A_\xi\, e_{m+\alpha} = -\lambda_\alpha e_\alpha && \text{for } \alpha = 1, \dots, m \\ &A_\xi e_0 = 0, \quad A_\xi e_\alpha = 0 && \text{for } \alpha = 2m+1, \dots, n. \end{aligned} \tag{10}$$

From the definition of the singular frame (4), we see that one may set

$$\begin{aligned} &f_\alpha = e_{m+\alpha}, \quad f_{m+\alpha} = -e_\alpha && \text{for } \alpha = 1, \dots, m \\ &f_0 = e_0, \quad f_\alpha = e_\alpha && \text{for } \alpha = 2m+1, \dots, n. \end{aligned} \tag{11}$$

A unit vector field ξ is called *normal* if $\big\langle R(X,Y)Z, \xi\big\rangle = 0$ and *strongly normal* if $\big\langle (\nabla_X A)Y, Z\big\rangle = 0$ for all $X, Y, Z \in \xi^\perp$ [7]. It is evident that each strongly normal vector field is always normal. If ξ is a unit *Killing* vector field the converse is also true.

Now we can essentially simplify the result of Lemma 1.2.

Lemma 2.1 *Let ξ be a Killing vector field on a Riemannian manifold M^{n+1}. Suppose that $\operatorname{rank} A_\xi = 2m$. Denote e_α ($\alpha = 1, \dots, n$) a singular frame for the field ξ satisfying (10) and (11). If ξ is (strongly) normal then the non-zero components of a second fundamental form of $\xi(M) \subset T_1M$ are given by*

$$\tilde\Omega_{\sigma|\, m+\sigma\, 0} = -\tilde\Omega_{m+\sigma|\, \sigma 0} = \frac{1}{2}\frac{K_\sigma(1-K_\sigma)}{1+K_\sigma} \qquad \sigma = 1, \dots, m,$$

where K_α are the sectional curvatures of M^{n+1} along the planes $\xi \wedge e_\alpha$.

Proof. Since ξ is a Killing vector field then [4]

$$A_\xi^* A_\xi X = R(X,\xi)\xi \tag{12}$$

and hence, the right singular vectors for Killing vector field are the Jacobi fields along ξ- geodesics. Let $e_1, \dots, e_n$ be an orthonormal frame of Jacobi fields. Then

$$R(e_\alpha, \xi)\xi = K_\alpha e_\alpha \quad \alpha = 1, \dots, n,$$

where K_α are the sectional curvatures of M along $\xi \wedge e_\alpha$ planes.

On the other hand, by definition of singular vectors

$$A_\xi^* A_\xi e_\alpha = \lambda_\alpha^2 e_\alpha.$$

Thus $K_\alpha = \lambda_\alpha^2 \ge 0$.

Any Killing vector field ξ satisfies

$$r(X,Y)\xi \overset{def}{=} -(\nabla_X A_\xi)Y = R(X,\xi)Y. \tag{13}$$

Therefore, $\tilde\Omega_{\sigma|00} = \langle r(e_0,e_0)\xi, f_\sigma\rangle = 0$. Next, from the relation

$$\lambda_\alpha\langle e_\sigma, f_\alpha\rangle = \langle e_\sigma, A_\xi e_\alpha\rangle = -\langle A_\xi e_\sigma, e_\alpha\rangle = -\lambda_\sigma\langle f_\sigma, e_\alpha\rangle$$

we find $\lambda_\alpha\lambda_\sigma\langle e_\sigma, f_\alpha\rangle = -\lambda_\sigma^2\,\langle f_\sigma, e_\alpha\rangle$ and therefore

$$\begin{aligned}
\tilde\Omega_{\sigma|\alpha 0} = \ & \tfrac12\Lambda_{\sigma\alpha 0}\Big\{\langle r(e_\alpha,e_0)\xi + r(e_0,e_\alpha)\xi, f_\sigma\rangle + \lambda_\sigma\lambda_\alpha\langle R(e_\sigma,e_0)\xi, f_\alpha\rangle\Big\} = \\
& \tfrac12\Lambda_{\sigma\alpha 0}\Big\{\langle R(e_\alpha,\xi)\xi, f_\sigma\rangle + \lambda_\sigma\lambda_\alpha\langle R(e_\sigma,\xi)\xi, f_\alpha\rangle\Big\} = \\
& \tfrac12\Lambda_{\sigma\alpha 0}\Big\{\lambda_\alpha^2\langle e_\alpha, f_\sigma\rangle + \lambda_\sigma^3\lambda_\alpha\langle e_\sigma, f_\alpha\rangle\Big\} = \\
& \tfrac12\Lambda_{\sigma\alpha 0}\Big\{\lambda_\alpha^2\langle e_\alpha, f_\sigma\rangle - \lambda_\sigma^4\langle e_\alpha, f_\sigma\rangle\Big\} = \\
& \tfrac12\tfrac{\lambda_\alpha^2-\lambda_\sigma^4}{\sqrt{(1+\lambda_\sigma^2)(1+\lambda_\alpha^2)}}\langle e_\alpha, f_\sigma\rangle.
\end{aligned}$$

Taking into account (11), we get

$$\tilde\Omega_{\sigma|\,m+\sigma\,0} = -\tilde\Omega_{m+\sigma|\,\sigma 0} = \frac12\frac{K_\sigma(1-K_\sigma)}{1+K_\sigma} \quad \sigma = 1,\dots,m. \tag{14}$$

Since ξ is strongly normal,

$$r(e_\alpha,e_\beta)\xi \sim \xi, \quad R(e_\alpha,e_\beta)\xi = 0$$

for all $\alpha,\beta = 1,\dots,n$. Therefore for the remain components we have $\tilde\Omega_{\sigma|\alpha\beta} = 0$.

■

One of the most important examples of Killing vector fields is a characteristic vector field of Sasakian structure. Using Lemma 2.1 we can prove the following.

Lemma 2.2 *Let M^{2m+1} be Sasakian manifold and ξ be a characteristic vector field. Then $\xi(M)$ is totally geodesic in T_1M.*

Proof. Let M^{2m+1} be an odd dimensional manifold admitting a unit vector field ξ, linear operator φ and 1-form η such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1$$

for any vector field X on M. A triple (φ,ξ,η) is called *an almost contact structure* on M and the manifold is called *an almost contact* manifold.

If the almost contact manifold is Riemannian with metric $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ and

$$\langle\varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y), \quad \eta(X) = \langle X, \xi\rangle$$

for any vector fields X and Y on M then a tetrad (φ, ξ, η, g) is called an almost contact *metric* structure and the manifold is called an almost contact *metric* manifold. The almost contact metric structure is said to be *contact* if $d\eta(X, Y) = \big\langle \varphi X, Y \big\rangle$ for all X, Y.

The vector field ξ is called *characteristic vector field* of almost contact metric structure. This field is always *geodesic*. If, in addition, ξ is a Killing vector field, then the almost contact metric manifold is called K-contact. If, moreover, the Riemannian curvature tensor R satisfies

$$R(X, Y)\,\xi = \big\langle \xi, Y \big\rangle X - \big\langle \xi, X \big\rangle Y. \tag{15}$$

for all vector fields X, Y on M, then the K-contact manifold is called Sasakian.

In Sasakian manifold

$$\nabla_X \xi = \varphi X, \quad (\nabla_X \varphi) Y = \big\langle \xi, Y \big\rangle X - \big\langle X, Y \big\rangle \xi = R(X, \xi) Y. \tag{16}$$

The property (15) implies that the sectional curvature of M along the planes involving characteristic vector field ξ is equal to 1 and ξ is a normal unit vector field while (16) means that ξ is strongly normal. So, we may apply Lemma 2.1 and immediately get $\Omega_{\sigma|\,\alpha\beta} \equiv 0$.

■

Now the sufficient part of the Theorem 2.1 follows immediately from Lemma 2.2. Remark, that for the 3-dimensional manifold the we can prove a stronger result.

Theorem 2.2 *Let ξ be a unit Killing vector field on 3-dimensional Riemannian manifold M^3. If $\xi(M^3)$ is totally geodesic in T_1M^3 then M^3 is either Sasakian and ξ is the characteristic vector field or $M^3 = M^2 \times E^1$ metrically and ξ is the unit vector field of Euclidean factor.*

Proof. Indeed, (14) is true for any unit Killing vector field. If $\xi(M^3)$ is totally geodesic, then there exist a Jacobi frame e_1, e_2 in $\xi^\perp$ such that the sectional curvatures $K_{\xi\wedge e_1} = K_{\xi\wedge e_2} = K$ and K satisfies $K(1-K) = 0$. Since e_1 and e_2 are Jacobi fields along ξ-geodesics, $\big\langle R(e_1, \xi)\xi, e_2 \big\rangle = 0$ and therefore, for any unit X in $\xi^\perp$ it is easy to find that $K_{\xi\wedge X} \equiv K$. Thus, all the sectional curvatures along 2-planes involving ξ is equal either 0 or 1. The first case means that ξ is a parallel vector field on M^3. So, $M^3 = M^2 \times E^1$ metrically and ξ is a unit vector field of Euclidean factor. The second case means that M^3 is K-contact which means in dimension 3 that M^3 is Sasakian and ξ is the characteristic vector field.

■

The necessary part of the proof. Suppose now that ξ is geodesic vector field generating a totally geodesic submanifold in T_1S^{n+1}.

Since ξ is geodesic vector field,

$$r(e_\alpha, \xi)\xi = \nabla_{e_\alpha}\nabla_\xi \xi - \nabla_{\nabla_{e_\alpha}\xi}\xi = -A_\xi^2 e_\alpha.$$

Since the manifold is of constant curvature 1,

$$\begin{aligned} &r(\xi, e_\alpha)\xi = R(\xi, e_\alpha)\xi + r(e_\alpha, \xi)\xi = -e_\alpha - A_\xi^2 e_\alpha, \\ &\lambda_\alpha\lambda_\sigma \big\langle R(e_\sigma, \xi)\xi, f_\alpha \big\rangle = \lambda_\alpha\lambda_\sigma \big\langle f_\alpha, e_\sigma \big\rangle = \big\langle A_\xi e_\alpha, A_\xi^* f_\sigma \big\rangle = \big\langle A_\xi^2 e_\alpha, f_\sigma \big\rangle. \end{aligned}$$

So we have

$$\tilde\Omega_{\sigma|\,\alpha 0} = \tfrac{1}{2}\Lambda_{\sigma\alpha 0}\left\{-2\left\langle A_\xi^2 e_\alpha, f_\sigma\right\rangle - \left\langle e_\alpha, f_\sigma\right\rangle + \left\langle A_\xi^2 e_\alpha, f_\sigma\right\rangle\right\} = -\tfrac{1}{2}\Lambda_{\sigma\alpha 0}\big\langle A_\xi^2 e_\alpha + e_\alpha, f_\sigma\big\rangle.$$

If ξ is a totally geodesic vector field, then

$$\big\langle A_\xi^2 e_\alpha + e_\alpha, f_\sigma\big\rangle = 0$$

for all $\alpha, \sigma = 1, \dots, n$. This means that

$$A_\xi^2 e_\alpha + e_\alpha = 0$$

for all α and therefore each e_α is the eigenvector for the operator A_ξ^2 corresponding to a common eigenvalue -1. Since A_ξ is a *real* operator, its eigenvalues are $\pm i$. In this case, with respect to some orthonormal frame, the matrix of $A_\xi|_{\xi^\perp}$ takes a box-diagonal form with boxes of type[4]

$$\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}.$$

This means that $A_\xi + A_\xi^* = 0$ and therefore, ξ is a Killing vector field on a sphere. Evidently, the dimension has to be be odd and the field has to be Hopf's one.

The proof is complete.

Remark. The Hopf vector field is totally geodesic on the *unit* sphere only. If $S^{2m+1}(r)$ is a sphere of radius r, then ξ is not Sasakian structure vector field but still unit Killing normal vector field satisfying $K_\sigma = \frac{1}{r}$ and therefore

$$\tilde\Omega_{\sigma|00} \equiv 0, \quad \tilde\Omega_{\sigma|\alpha\beta} \equiv 0,$$
$$\tilde\Omega_{\sigma|m+\sigma 0} = -\tilde\Omega_{m+\sigma|\sigma 0} = \frac{1}{2\,(1+\frac{1}{r^2})^{3/2}}\,\frac{1}{r^2}(1-\frac{1}{r^2}) \quad (\alpha, \beta, \sigma = 1, \dots, m).$$

Thus, if $r \neq 1$ then ξ does not generate a totally geodesic submanifold. Nevertheless, it stays minimal.

3 Stability of the Hopf vector field.

3.1 General formula and preparations.

Let M^n be a submanifold in a Riemannian space R^m. Denote by $\nabla^\perp$ a covariant derivative in normal bundle connection. Let e_i $(i = 1, \dots, n)$ be an orthonormal frame on M^n and η be a normal variation field along M^n. Denote by $k_i(\eta)$ an i-th principal normal curvature of M^n with respect to η and $K(e_i, \eta)$ the sectional curvature of R^m along a plane $e_i \wedge \eta$. Then *the volume second variation of M^n with respect to η* is given by [3]

$$\delta^2 Vol(\eta) = \int\limits_{M^n} \left\{ \sum_{i=1}^n \big\langle \nabla^\perp_{e_i}\eta, \nabla^\perp_{e_i}\eta \big\rangle - |\eta|^2 \Big(-\sum_{i,j;\, i\neq j} k_i(\eta)k_j(\eta) + \sum_{i=1}^n K(e_i, \eta) \Big) \right\} dV. \tag{17}$$

[4]With additional condition of covariant normality (see the footnote on page 4)

To apply the formula (17) to our case we should find the normal bundle connection for the submanifold $\xi(S^{2m+1})\subset T_1S^{2m+1}$.

Lemma 3.1 *Let X and Y be arbitrary vector fields on S^{n+1} ($n=2m$). Denote $Y^\perp = Y-\langle\xi,Y\rangle\xi$. Then, with respect to the Hopf unit vector field ξ,*

$$\bar\nabla_{X^\tau_\xi}Y^\nu_\xi = \big(\nabla_X Y^\perp\big)^\nu_\xi - \frac{1}{2}\langle\xi,X\rangle\big(A_\xi Y^\perp\big)^\nu_\xi,$$

where $\bar\nabla$ means a covariant derivative with respect to the Levi-Civita connection on T_1S^n.

Proof. With respect to the Levi-Civita connection $\bar\nabla$ on T_1M we have [1]

$$\bar\nabla_{X^h}Y^h = (\nabla_XY)^h - \frac{1}{2}\big(R(X,Y)\xi\big)^t,\quad \bar\nabla_{X^h}Y^t = (\nabla_XY)^t + \frac{1}{2}\big(R(\xi,Y)X\big)^h,$$

$$\bar\nabla_{X^t}Y^h = \frac{1}{2}\big(R(\xi,X)Y\big)^h,\qquad \bar\nabla_{X^t}Y^t = -\langle Y,\xi\rangle X^t.$$

Using this formulas, we get

$$\begin{aligned}\bar\nabla_{X^\tau_\xi}Y^\nu_\xi &= \bar\nabla_{X^h}(A^*_\xi Y)^h + \bar\nabla_{X^h}Y^t - \bar\nabla_{(A^*_\xi X)^t}(A^*_\xi Y)^h - \bar\nabla_{(A^*_\xi X)^t}Y^t = \\ &\left[\nabla_X(A_\xi Y) + \frac{1}{2}R(\xi,Y)X - \frac{1}{2}R(\xi,A_\xi X)A^*_\xi Y\right]^h + \\ &\left[\nabla_XY - \frac{1}{2}R(X,A^*_\xi Y)\xi + \langle Y,\xi\rangle A_\xi X\right]^t.\end{aligned}$$

Since ξ is a Killing vector field, (9) and (13) are fulfilled and we have

$$\begin{aligned}\bar\nabla_{X^\tau_\xi}Y^\nu_\xi &= \big[-(\nabla_XA_\xi)Y - A_\xi(\nabla_XY) + \tfrac{1}{2}R(\xi,Y)X + \tfrac{1}{2}R(\xi,A_\xi X)A_\xi Y\big]^h + \\ &\big[\nabla_XY + \tfrac{1}{2}R(X,A_\xi Y)\xi + \langle Y,\xi\rangle A_\xi X\big]^t = \\ &\big[R(X,\xi)Y + \tfrac{1}{2}R(\xi,Y)X + \tfrac{1}{2}R(\xi,A_\xi X)A_\xi Y\big]^h + \\ &\big[\tfrac{1}{2}R(X,A_\xi Y)\xi + \langle Y,\xi\rangle A_\xi X\big]^t + (\nabla_XY)^\nu_\xi.\end{aligned}$$

Since R is a curvature tensor of a unit sphere and keeping in mind (9) and (12), we continue

$$\begin{aligned}\bar\nabla_{X^\tau_\xi}Y^\nu_\xi &= \big[\langle\xi,Y\rangle X - \tfrac{1}{2}\langle X,Y\rangle\xi - \tfrac{1}{2}\langle\xi,X\rangle Y + \tfrac{1}{2}\langle A_\xi X,A_\xi Y\rangle\xi\big]^h + \\ &\big[-\tfrac{1}{2}\langle\xi,X\rangle A_\xi Y + \langle\xi,Y\rangle A_\xi X\big]^t + (\nabla_XY)^\nu_\xi = \\ &\big[\langle\xi,Y\rangle X - \tfrac{1}{2}\langle X,Y\rangle\xi - \tfrac{1}{2}\langle\xi,X\rangle Y + \tfrac{1}{2}\langle X,Y\rangle\xi - \\ &\tfrac{1}{2}\langle\xi,X\rangle\langle\xi,Y\rangle\xi\big]^h + \big[-\tfrac{1}{2}\langle\xi,X\rangle A_\xi Y + \langle\xi,Y\rangle A_\xi X\big]^t + (\nabla_XY)^\nu_\xi = \\ &\big[\langle\xi,Y\rangle\big(X-\langle\xi,X\rangle\xi\big) - \tfrac{1}{2}\langle\xi,X\rangle\big(Y-\langle\xi,Y\rangle\xi\big)\big]^h + \\ &\big[-\tfrac{1}{2}\langle\xi,X\rangle A_\xi Y + \langle\xi,Y\rangle A_\xi X\big]^t + (\nabla_XY)^\nu_\xi = \\ &\Big[\langle\xi,Y\rangle\big(A^*_\xi A_\xi X\big) - \tfrac{1}{2}\langle\xi,X\rangle\big(A^*_\xi A_\xi Y\big)\Big]^h + \\ &\big[-\tfrac{1}{2}\langle\xi,X\rangle A_\xi Y + \langle\xi,Y\rangle A_\xi X\big]^t + (\nabla_XY)^\nu_\xi = \\ &(\nabla_XY)^\nu_\xi + \langle\xi,Y\rangle\big(A_\xi X\big)^\nu_\xi - \tfrac{1}{2}\langle\xi,X\rangle\big(A_\xi Y\big)^\nu_\xi.\end{aligned}$$

Consider, now, $\big(\nabla_X Y^\perp)^\nu_\xi$.

$$\begin{aligned}(\nabla_X Y^\perp)^\nu_\xi = \; & \big[A^*_\xi \nabla_X Y^\perp\big]^h + \big[\nabla_X Y^\perp\big]^t = \\ & \Big[A^*_\xi \nabla_X\big(Y - \langle\xi, Y\rangle\xi\big)\Big]^h + \big[\nabla_X\big(Y - \langle\xi, Y\rangle\xi\big)\big]^t = \\ & \Big[A^*_\xi \nabla_X Y + \langle\xi, Y\rangle A^*_\xi A_\xi X\Big]^h + \big[\nabla_X Y + \langle\xi, Y\rangle A_\xi X\big]^t = \\ & (\nabla_X Y)^\nu_\xi + \langle\xi, Y\rangle\big(A_\xi X\big)^\nu_\xi.\end{aligned}$$

Since $A_\xi\,\xi = 0$, we see that $A_\xi Y = A_\xi Y^\perp$. Comparing the results, we get

$$\bar\nabla_{X^\tau_\xi} Y^\nu_\xi = \big(\nabla_X Y^\perp\big)^\nu_\xi - \frac{1}{2}\langle\xi, X\rangle\big(A_\xi Y^\perp\big)^\nu_\xi$$

what was claimed.

■

Remark. As we see, the tangent component of the latter derivative is zero, which gives *another proof* of totally geodesic property of the Hopf vector field.

Since $Y^\nu_\xi = (Y^\perp)^\nu_\xi$, we may consider only the vectors from $\xi^\perp$ in all ξ-normal lifting operations. So, keep this in mind in what follows.

Corollary 3.1 *Let $\bar\nabla^\perp$ denote a covariant derivative in normal bundle of $\xi(M)$. If ξ is a Hopf vector field on the unit sphere S^{n+1} $(n = 2m)$ then*

$$\bar\nabla^\perp_{X^\tau_\xi} Y^\nu_\xi = \big(\nabla_X Y\big)^\nu_\xi - \frac{1}{2}\langle\xi, X\rangle\big(A_\xi Y\big)^\nu_\xi = -\langle\xi, X\rangle Y^h + 2\big(\nabla_X Y\big)^t, \qquad (18)$$

for any X and $Y \in \xi^\perp$.

Proof. Indeed, if ξ is a Hopf vector field for any Y, Z we have

$$\begin{aligned}\langle\langle Y^\nu_\xi, Z^\nu_\xi\rangle\rangle = \; & \langle A^*_\xi Y, A^*_\xi Z\rangle + \langle Y, Z\rangle - \langle\xi, Z\rangle\langle\xi, Y\rangle = \\ & 2\langle Y, Z\rangle - 2\langle\xi, Z\rangle\langle\xi, Y\rangle.\end{aligned}$$

Let $Y, Z \in \xi^\perp$. Then by Lemma 3.1 and (9)

$$\begin{aligned}& \langle\langle\bar\nabla_{X^\tau_\xi} Y^\nu_\xi, Z^\nu_\xi\rangle\rangle = \langle\langle\big(\nabla_X Y\big)^\nu_\xi - \tfrac{1}{2}\langle\xi, X\rangle\big(A_\xi Y\big)^\nu_\xi, Z^\nu_\xi\rangle\rangle = \\ & 2\langle\nabla_X Y - \tfrac{1}{2}\langle\xi, X\rangle A_\xi Y, Z\rangle - 2\langle\nabla_X Y, \xi\rangle\langle\xi, Z\rangle = \\ & 2\langle\nabla_X Y - \langle\nabla_X Y, \xi\rangle\xi, Z\rangle + \langle\xi, X\rangle\langle Y, A_\xi Z\rangle = \\ & \langle\langle 2\,(\nabla_X Y)^t, Z^t\rangle\rangle - \langle\langle\,\langle\xi, X\rangle Y^h, -(A_\xi Z)^h\rangle\rangle \\ & \langle\langle -\langle\xi, X\rangle Y^h + 2(\nabla_X Y)^t, (A^*_\xi Z)^h + Z^t\rangle\rangle = \\ & \langle\langle -\langle\xi, X\rangle Y^h + 2(\nabla_X Y)^t, Z^\nu_\xi\rangle\rangle.\end{aligned}$$

Since Z is arbitrary, we get the result.

■

3.2 Second variation of the volume for the Hopf vector field.

Let ξ be a given unit vector field on M^{n+1}. Then the vector field $\tilde\eta = (\eta)^\nu_\xi$ can be considered as a field of volume variation for the submanifold $\xi(M^{n+1})$, where η is an arbitrary vector field in $\xi^\perp$. For the case of the Hopf vector field on S^{n+1} $(n = 2m)$, we may choose

$$\tilde e_i = \frac{1}{||(e_i)^\tau_\xi||}(e_i)^\tau_\xi = \begin{cases} e^h_0 \ \text{ for } i = 0\ , \\ \frac{1}{\sqrt2}\big(e^h_\alpha + f^t_\alpha\big) \ \text{ for } \alpha = 1,\dots,n, \end{cases} \tag{19}$$

where $e_0, e^1, \dots, e_n,\ f_0, f_1, \dots, f_n$ form the singular bases for the A_ξ- operator, as an orthonormal tangent frame of $\xi(S^{n+1})$. Since $\xi(S^{n+1})$ is totally geodesic, the Duschek formula (17) obtains the form

$$\delta^2 Vol(\tilde\eta) = \int\limits_{S^{n+1}} \left\{\sum_{i=0}^{n} \|\nabla^\perp_{\tilde e_i}\tilde\eta\| - \|\tilde\eta\|^2 \sum_{i=0}^{n} \tilde K(\tilde e_i, \tilde\eta)\right\} dV, \tag{20}$$

where $\bar\nabla^\perp_{\tilde e_i}$ is given by (18) and $\tilde K$ is given by [2]

$$\begin{aligned}\tilde K(\tilde X, \tilde Y) \ &= \big\langle R(X_1,Y_1)Y_1, X_1\big\rangle - \tfrac34 \,\big|\, R(X_1,Y_1)\xi\,\big|^2 + \\ &\tfrac14 \,\big|\, R(\xi,Y_2)X_1 + R(\xi,X_2)Y_1\big|^2 + |X_2|^2|Y_2|^2 - \big\langle X_2, Y_2\big\rangle^2 + \\ &3\big\langle R(X_1,Y_1)Y_2, X_2\big\rangle - \big\langle R(\xi,X_2)X_1, R(\xi,Y_2)Y_1\big\rangle + \\ &\big\langle(\nabla_{X_1}R)(\xi,Y_2)Y_1, X_1\big\rangle + \big\langle(\nabla_{Y_1}R)(\xi,X_2)X_1, Y_1\big\rangle,\end{aligned} \tag{21}$$

for an orthonormal basis $\tilde X = X^h_1 + X^v_2$, $\tilde Y = Y^h_1 + Y^v_2$ ($X_2, Y_2 \in \xi^\perp$) of a 2-plane tangent to T_1S^{n+1}.

Lemma 3.2 *Let ξ be the Hopf vector field on S^{n+1} $(n = 2m)$. Let $\eta \in \xi^\perp$ generates a normal volume variation $\tilde\eta = (\eta)^\nu_\xi$ for the submanifold $\xi(S^{n+1}) \subset T_1(S^{n+1})$. Let $e_0 = \xi, e_1, \dots, e_n$ is a right singular frame for the operator A_ξ. Then, the Duschek formula (17) can be reduced to the form*

$$\delta^2 Vol(\tilde\eta) = \int\limits_{S^{n+1}} \left\{4|\nabla_{e_0}\eta|^2 + 2\sum_{\alpha=1}^{n} |\nabla_{e_\alpha}\eta|^2 - \frac{2n-1}{2}\,|\eta|^2\right\} dV, \tag{22}$$

Proof. Denote $\bar\nabla^\perp$ the normal bundle connection for $\xi(S^{n+1})$. Prove, first, that

$$\sum_{i=0}^{n} \|\bar\nabla^\perp_{\tilde e_i}\tilde\eta\| = 4|\nabla_{e_0}\eta|^2 + 2\sum_{\alpha=1}^{n} |\nabla_{e_\alpha}\eta|^2 - |\eta|^2.$$

Applying (18) and keeping in mind (19), we find

$$\bar\nabla^\perp_{\tilde e_0}\tilde\eta = -(\eta)^h + 2(\nabla_{e_0}\eta)^t$$
$$\bar\nabla^\perp_{\tilde e_\alpha}\tilde\eta = \sqrt2\,(\nabla_{e_\alpha}\eta)^t$$

Therefore

$$\begin{aligned}\sum_{i=0}^{n}\|\bar\nabla^{\perp}_{\tilde e_i}\tilde\eta\,\| = \;& |\eta\,|^2+4\,|\nabla_{e_0}\eta\,|^2+2\textstyle\sum_{\alpha=1}^{n}\Big(|\nabla_{e_\alpha}\eta\,|^2-\big\langle\nabla_{e_\alpha}\eta\,,\xi\big\rangle^2\Big)=\\ & |\eta\,|^2+4\,|\nabla_{e_0}\eta\,|^2+2\textstyle\sum_{\alpha=1}^{n}\Big(|\nabla_{e_\alpha}\eta\,|^2-\big\langle\eta\,,\nabla_{e_\alpha}\xi\big\rangle^2\Big)=\\ & |\eta\,|^2+4\,|\nabla_{e_0}\eta\,|^2+2\textstyle\sum_{\alpha=1}^{n}\,|\nabla_{e_\alpha}\eta\,|^2-2\textstyle\sum_{\alpha=1}^{n}\big\langle\eta\,,f_\alpha\big\rangle^2=\\ & |\eta\,|^2+4\,|\nabla_{e_0}\eta\,|^2+2\textstyle\sum_{\alpha=1}^{n}\,|\nabla_{e_\alpha}\eta\,|^2-2\,|\eta\,|^2=\\ & 4\,|\nabla_{e_0}\eta\,|^2+2\textstyle\sum_{\alpha=1}^{n}\,|\nabla_{e_\alpha}\eta\,|^2-|\eta\,|^2\end{aligned}$$

and the proof is complete.

Now prove that

$$\|\tilde\eta\,\|^2\sum_{i=0}^{n}\tilde K(\tilde e_i,\tilde\eta\,)=\frac{1}{2}|\eta\,|^2+(n-2)\,|\eta\,|^2.$$

The sectional curvature of T_1M^n is given by (21) in presumption that $\tilde X$ and $\tilde Y$ are orthonormal. To find $\|\tilde\eta\,\|^2\tilde K(\tilde e_i,\tilde\eta\,)$ we can use (21) setting $\tilde Y=\tilde\eta$ and keeping in mind that $\tilde\eta$ is of arbitrary length.

Now set $\zeta=A_\xi\eta$ and then $\tilde\eta=\zeta^h+\eta^t$. Evidently,

$$\big\langle\eta,\zeta\big\rangle=0,\quad\big\langle\eta,\xi\big\rangle=\big\langle\zeta,\xi\big\rangle=0,\quad|\eta\,|=|\zeta|. \tag{23}$$

Set $\tilde X=\tilde e_0=e_0^h$ and $\tilde Y=\zeta^h+\eta^t$. Then

$$\begin{aligned}\|\tilde\eta\,\|^2\tilde K(\tilde e_0,\tilde\eta\,)= \;& \big\langle R(e_0,\zeta)\zeta,e_0\big\rangle-\frac{3}{4}\big|\,R(e_0,\zeta)\xi\,\big|^2+\frac{1}{4}\big|\,R(\xi,\eta)e_0\,\big|^2=\\ & |\zeta|^2-\frac{3}{4}\,|\zeta|^2+\frac{1}{4}\,|\eta\,|^2=\frac{1}{2}\,|\eta\,|^2\end{aligned}$$

To find $\tilde K(\tilde e_\alpha,\tilde\eta\,)$ for $\alpha=1,\dots,n$ we set $\tilde X=\frac{1}{\sqrt2}(e_\alpha^h+f_\alpha^t)$ and therefore

$$X_1=\frac{1}{\sqrt2}e_\alpha,\;X_2=\frac{1}{\sqrt2}f_\alpha,\;Y_1=\zeta,\;Y_2=\eta$$

in application of (21). Thus we have

$$\begin{aligned}\|\tilde\eta\,\|^2\tilde K(\tilde e_\alpha,\tilde\eta)\; & =\frac{1}{2}\big\langle R(e_\alpha,\zeta)\zeta,e_\alpha\big\rangle-\frac{3}{8}\big|\,R(e_\alpha,\zeta)\xi\,\big|^2+\\ & \frac{1}{8}\big|\,R(\xi,\eta)e_\alpha+R(\xi,f_\alpha)\zeta\big|^2+\frac{1}{2}|\eta\,|^2-\frac{1}{2}\big\langle f_\alpha,\eta\big\rangle^2+\\ & \frac{3}{2}\big\langle R(e_\alpha,\zeta)\eta,f_\alpha\big\rangle-\frac{1}{2}\big\langle R(\xi,f_\alpha)e_\alpha,R(\xi,\eta)\zeta\big\rangle.\end{aligned}$$

Now set $\eta^\alpha=\big\langle\eta,f_\alpha\big\rangle$. Then $\zeta^\alpha=\big\langle\zeta,e_\alpha\big\rangle=-\big\langle\eta,f_\alpha\big\rangle=-\eta^\alpha$. Keeping in mind (23) we get

$$\begin{aligned}\|\tilde\eta\,\|^2\tilde K(\tilde e_\alpha,\tilde\eta)= \;& \frac{1}{2}\,|\eta\,|^2-\frac{1}{2}(\eta^{\,\alpha})^2+\frac{1}{8}\,\big|\big\langle e_\alpha,\eta\,\big\rangle+\big\langle f_\alpha,\zeta\,\big\rangle\big|^2+\frac{1}{2}|\eta\,|^2-\\ & \tfrac{1}{2}(\eta^{\,\alpha})^2-\tfrac{3}{2}\big\langle e_\alpha,\eta\,\big\rangle\big\langle f_\alpha,\zeta\,\big\rangle=|\eta\,|^2-(\eta^{\,\alpha})^2-\big\langle\eta,e_\alpha\big\rangle^2.\end{aligned}$$

It is also easy to see that

$$\sum_{\alpha=1}^{n}(\eta^{\alpha})^2 = \sum_{\alpha=1}^{n}\langle\eta, e_\alpha\rangle^2 = |\eta|^2.$$

Hence we have

$$\|\tilde\eta\|^2\left(\tilde K(\tilde e_0,\tilde\eta)+\sum_{\alpha=0}^{n}\tilde K(\tilde e_\alpha,\tilde\eta)\right)=\frac{1}{2}|\eta|^2+(n-2)|\eta|^2.$$

Combining the results we get what was claimed.

■

Proposition 3.1 *The Hopf vector field on the unit 3-sphere is stable.*

Proof. Since the subspace of right (and left) singular frames for the Hopf vector field coincides with $\xi^\perp$, we may choose the other two Hopf vector fields on S^3 as e_1 and e_2. Then

$$\begin{array}{ll}\nabla_{e_1}e_1=0, & \nabla_{e_1}e_2=-e_0,\\ \nabla_{e_2}e_1=e_0, & \nabla_{e_2}e_2=0.\end{array}\tag{24}$$

Set $\eta=\eta^1e_1+\eta^2e_2$. Set $|\mathrm{grad}\,\eta^{\alpha}|^2=\big(e_1(\eta^\alpha)\big)^2+\big(e_2(\eta^\alpha)\big)^2$ $(\alpha=1,2)$. Then, using (24) we find

$$\sum_{\alpha=1}^{2}|\nabla_{e_\alpha}\eta|^2=\sum_{\alpha=1}^{n}|\,\mathrm{grad}\,\eta^\alpha|^2+|\eta|^2.$$

Therefore,

$$\delta^2 Vol(\tilde\eta)=\int\limits_{S^{n+1}}\left\{4|\nabla_{e_0}\eta|^2+2\sum_{\alpha=1}^{n}|\mathrm{grad}\,\eta^\alpha|^2+\frac{1}{2}|\eta|^2\right\}dV>0$$

which means that $\xi(S^3)$ is stable.

■

Proposition 3.2 *The Hopf vector field on the unit n-sphere for $n=2m+1>3$ is unstable.*

Proof. Choose the vectors of the singular frame such that $e_0=\xi$ while the other $e_1,\dots,e_{2m}$ are the horizontal lifts of vectors of orthonormal frame q_k of CP^m with respect to the Hopf fibration $S^{2m+1}\xrightarrow{S^1}CP^m$. Then $\nabla_{q_k}q_j=0$ and $Jq_{2k}=q_{2k+1}$ $(k,j=1,\dots,m)$ for complex structure of CP^n (see [8]). Then along the fiber, i.e. along the integral curves of $e_0=\xi$, the following table of non-zero covariant derivatives can be achieved [8, 10].

$$\begin{array}{ll}\nabla_{e_0}e_{2k}=e_{2k-1}, & \nabla_{e_0}e_{2k-1}=-e_{2k},\\ \nabla_{e_{2k}}e_0=e_{2k-1}, & \nabla_{e_{2k-1}}e_0=-e_{2k}\\ \nabla_{e_{2k-1}}e_{2k}=e_0, & \nabla_{e_{2k}}e_{2k-1}=-e_0,\end{array}$$

where $k = 1, \dots, m$.

Let $\eta = \eta^{2k-1} e_{2k-1} + \eta^{2k} e_{2k}$ be a variation field. Then, using a table of derivatives, we find

$$
\begin{aligned}
\nabla_{e_0} \eta &= \big(e_0(\eta^{2k-1}) + \eta^{2k}\big)\, e_{2k-1} + \big(e_0(\eta^{2k}) - \eta^{2k-1}\big)\, e_{2k-1} \\
\nabla_{e_{2t-1}} \eta &= e_{2t-1}(\eta^{2k-1})\, e_{2k-1} + e_{2t-1}(\eta^{2k})\, e_{2k} + \delta_{st}\, \eta^{2k}\, e_0 \\
\nabla_{e_{2t}} \eta &= e_{2t}(\eta^{2k-1})\, e_{2k-1} + e_{2t}(\eta^{2k})\, e_{2k} - \delta_{kt}\, \eta^{2k-1}\, e_0,
\end{aligned}
$$

where δ_{kt} is the Kronecker symbol.

Set

$$|\mathrm{grad}\, \eta^{\sigma}|^2 = \sum_{\alpha=1}^{n} \big[e_\alpha(\eta^{\sigma})\big]^2.$$

Then

$$\sum_{\alpha=1}^{n} |\nabla_{e_\alpha} \eta\,|^2 = \sum_{\sigma=1}^{n} |\mathrm{grad}\, \eta^{\sigma}|^2 + |\eta\,|^2$$

and

$$|\nabla_{e_0} \eta\,|^2 = \sum_{k=1}^{m} \Big[\big(e_0(\eta^{2k}) - \eta^{2k-1}\big)^2 + \big(e_0(\eta^{2k-1}) + \eta^{2k}\big)^2\Big].$$

To prove the instability, we should find variation field η providing a negative sign for the second volume variation. So, choose

$$\eta = \cos t e_{2k-1} + \sin t e_{2k},$$

where t is an arc-length parameter on e_0-curves. Then

$$\nabla_{e_0} \eta = 0, \quad \mathrm{grad}\, \eta^{\sigma} = 0$$

and for the integrand in the Duscheck formula (22) we have

$$\frac{5 - 2n}{2} |\eta\,|^2 < 0$$

for $n > 2$, which completes the proof.

■

4 Sectional curvature of the Hopf vector field.

Since the Hopf vector field is totally geodesic in $T_1 S^{n+1}$, the sectional curvature of $\xi(S^{n+1})$ is completely defined by the curvature of $T_1 S^{n+1}$ along the planes, tangent to $\xi(S^{n+1})$. We can easily find it applying (3) and (21) to the Hopf vector field. Remind that the Hopf vector field is a characteristic one for the contact metric structure on the spheres. In contact metric geometry, the sections, containing characteristic vector field, are called ξ-sections, while the sections of type $X \wedge \varphi X$ for $X \in \xi^{\perp}$ are called φ-sections. Using the notion of ξ-tangential lift (3), for any $X \in \xi^{\perp}$ we call a 2-plane $X^{\tau}_{\xi} \wedge (\xi)^{\tau}_{\xi}$ as a ξ-tangential lift of ξ-section and a 2-plane $X^{\tau}_{\xi} \wedge (\nabla_X \xi)^{\tau}_{\xi}$, as a ξ-tangential lift of φ-section. The following assertion holds.

Theorem 4.1 *The sectional curvature of $\xi(S^{2m+1})$ for the Hopf vector field varies between $\frac{1}{4}$ and $\frac{5}{4}$. The curvature is minimal for ξ-tangential lift of ξ-section and maximal for ξ-tangential lift of φ-section*

The proof is elementary consequence of the Proposition below. It should be mentioned that the curvature of T_1S^{n+1} varies between 0 and 5/4 [13]. The theorem clarifies geometrical meaning of the maximal curvature. The minimal curvature is, geometrically, the ξ-sectional curvature of natural Sasakian structure on T_1S^{n+1} (which,as well known, equals to 1/4 after rescaling).

Proposition 4.1 *Let ξ be the Hopf vector field on S^{n+1} $(n = 2m)$. Let X^τ_ξ, Y^τ_ξ be ξ-tangential lifts of orthonormal vectors X and Y respectively. The sectional curvature $\tilde K(X^\tau_\xi, Y^\tau_\xi)$ of $\xi(S^{n+1})$ along the 2-plane $(X^\tau_\xi,\ Y^\tau_\xi)$ is given by*

$$\tilde K(X^\tau_\xi, Y^\tau_\xi) = \frac{1 - \dfrac{3}{4}\big[\,\langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2\,\big] + \dfrac{3}{2}\,\langle A_\xi X, Y\rangle^2}{2 - \big[\,\langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2\,\big]}$$

Proof. Let X and Y be *unit mutually orthogonal* vector fields on S^{n+1}. Then $X^\tau_\xi = X^h - (A_\xi X)^t$ and $Y^\tau_\xi = Y^h - (A_\xi Y)^t$ form a basis of elementary 2-plane, tangent to $\xi(S^{n+1})$. This basis is not orthonormal, since

$$\begin{aligned}
&\|X^\tau_\xi\|^2 = |X|^2 + |A_\xi X|^2 = 2 - \langle \xi, X\rangle^2,\\
&\|Y^\tau_\xi\|^2 = |Y|^2 + |A_\xi Y|^2 = 2 - \langle \xi, Y\rangle^2,\\
&\langle\langle X^\tau_\xi, Y^\tau_\xi\rangle\rangle = \langle X, Y\rangle + \langle A_\xi X, A_\xi Y\rangle = -\langle \xi, X\rangle\langle \xi, Y\rangle.
\end{aligned}$$

Therefore, the norm of bivector $X^\tau_\xi \wedge Y^\tau_\xi$ is

$$\|X^\tau_\xi \wedge Y^\tau_\xi\| = 4 - 2\,\langle \xi, X\rangle^2 - 2\,\langle \xi, Y\rangle^2. \tag{25}$$

Now set $X_1 = X,\ Y_1 = Y,\ X_2 = A_\xi X,\ Y_2 = A_\xi Y$ and apply (21). We have

$$\begin{aligned}
&\langle R(X_1, Y_1)Y_1, X_1\rangle = 1,\\
&|\,R(X_1, Y_1)\xi\,|^2 = |\langle \xi, Y\rangle X - \langle \xi, X\rangle Y|^2 = \langle \xi, Y\rangle^2 + \langle \xi, Y\rangle^2,\\
&|\,R(\xi, Y_2)X_1 + R(\xi, X_2)Y_1|^2 = |\langle A_\xi Y, X\rangle\xi - \langle \xi, X\rangle A_\xi Y +\\
&\langle A_\xi X, Y\rangle\xi - \langle \xi, Y\rangle A_\xi X|^2 = \langle \xi, X\rangle^2 |A_\xi Y|^2 + \langle \xi, Y\rangle^2 |A_\xi X|^2 +\\
&2\,\langle \xi, X\rangle\langle \xi, Y\rangle\langle A_\xi X, A_\xi Y\rangle = \langle \xi, X\rangle^2(1 - \langle \xi, Y\rangle^2\,) + \langle \xi, Y\rangle^2(1 - \langle \xi, X\rangle^2\,) -\\
&2\,\langle \xi, X\rangle^2\langle \xi, Y\rangle^2 = \langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2 - 4\,\langle \xi, X\rangle^2\langle \xi, Y\rangle^2,\\
&|X_2|^2|Y_2|^2 - \langle X_2, Y_2\rangle^2 = |A_\xi X|^2|A_\xi Y|^2 - \langle A_\xi X, A_\xi Y\rangle^2 =\\
&(1 - \langle \xi, X\rangle^2\,)(1 - \langle \xi, Y\rangle^2\,) - \langle \xi, X\rangle^2\langle \xi, Y\rangle^2 = 1 - \langle \xi, X\rangle^2 - \langle \xi, Y\rangle^2,\\
&\langle R(X_1, Y_1)Y_2, X_2\rangle = \langle\, -\,\langle A_\xi Y, X\rangle\,Y, A_\xi X\rangle = \langle A_\xi X, Y\rangle^2,\\
&\langle R(\xi, X_2)X_1, R(\xi, Y_2)Y_1\rangle = \langle\,\langle \xi, X\rangle\,A_\xi X, \langle \xi, Y\,\rangle\,A_\xi Y\rangle = -\langle \xi, X\rangle^2\langle \xi, Y\rangle^2.
\end{aligned}$$

Substituting the latter equalities into (21) and dividing the result by (25), we get

$$\tilde{K}(X^\tau_\xi, Y^\tau_\xi) = \frac{1 - \frac{3}{4}\big[\,\langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2\,\big] + \frac{3}{2}\,\langle A_\xi X, Y\rangle^2}{2 - \big[\,\langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2\,\big]}$$

■

References

[1] Boeckx E.,Vanhecke L. Characteristic reflections on unit tangent sphere bundle. *Houston J. Math.*, 23 (1997), 427 – 448.

[2] Borisenko A., Yampolsky A. The sectional curvature of the Sasaki metric of T_1M^n. *Ukr. Geom. Sb.*, 30 (1987), 10 – 17. (Enl. transl.: *J. Sov. Math.*, 51 (1990), 5, 2503 – 2508).

[3] Duschek A. Zur geometrischen Variationsrechnung. *Math. Z.,* 40 (1936), 279 – 374.

[4] Gil-Medrano O.,Llinares-Fuster E. Minimal unit vector fields. *Tôhoku Math. J.*, 54 (2002), 71 – 84.

[5] Gil-Medrano O.,Llinares-Fuster E. Second variation of volume and energy of vector fields. Stability of Hopf vector fields. *Math. Ann.* 320 (2001), 531 – 545.

[6] Gluck H.,Ziller W. On the volume of a unit vector field on the three-sphere, *Comm. Math. Helv.* 61 (1986), 177 – 192.

[7] González-Dávila J.C., Vanhecke L. Examples of minimal unit vector fields. *Ann Global Anal. Geom.* 18 (2000), 385 – 404.

[8] Han D.-S., Yim J.-W. Unit vector fields on spheres, which are harmonic maps. *Math. Z.*, 227 (1998), 83 – 92.

[9] Horn R., Jonson Ch. Matrix Analysis. *Cambridge Univ. Press, Cambridge, 1986.*

[10] Jonson D.L. Volume of flows, *Proc. Amer. Math. Soc.*, 104 (1988), 3, 923 – 931.

[11] Pedersen L.S. Volumes of vector fields on spheres, *Trans. Amer. Math. Soc.*, 336 (1993), 69 – 78.

[12] Yampolsky A. On the mean curvature of a unit vector field, *Math. Publ. Debrecen*, 60/1-2 (2002), 131 – 155.

[13] Yampolsky A. On the curvature of Sasaki metric of tangent sphere bundle. *Ukr. Geom. Sb.* 28 (1985), 132 – 145 (Russian) (Eng. transl.: *Journ. Sov. Math.* 48 (1990), 1, 108 – 117).

[14] Yampolsky A. On the intrinsic geometry of a unit vector field. *Comment. Math. Univ. Carolinae* 43, 2 (2002), 299 – 317.

Department of Geometry,
Faculty of Mechanics and Mathematics,
Kharkiv National University,
Svobody Sq. 4,
61077, Kharkiv,
Ukraine.
e-mail: yamp@univer.kharkov.ua