Seventh International Conference on
Geometry, Integrability and Quantization
June 2–10, 2005, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel De Leon, Editors
SOFTEX, Sofia 2005, pp 1–15

ON SPECIAL TYPES OF MINIMAL AND TOTALLY GEODESIC UNIT VECTOR FIELDS

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Abstract. We present a new equation with respect to a unit vector field on Riemannian manifold M^n such that its solution defines a totally geodesic submanifold in the unit tangent bundle with Sasaki metric and apply it to some classes of unit vector fields. We introduce a class of covariantly normal unit vector fields and prove that within this class the Hopf vector field is a unique global one with totally geodesic property. For the wider class of geodesic unit vector fields on a sphere we give a new necessary and sufficient condition to generate a totally geodesic submanifold in T_1S^n.

Key words: *Sasaki metric, minimal unit vector field, totally geodesic unit vector field, strongly normal unit vector field, Sasakian space form.*

AMS subject class: Primary 53B20, 53B25; Secondary 53C25.

1. Introduction

This paper is organized as follows. In Section 2 we give definitions of harmonic and minimal unit vector fields, rough Hessian and harmonicity tensor for the unit vector field. In Section 3 we give definition of a totally geodesic unit vector field and prove a basic Lemma 2 which gives a necessary and sufficient condition for the unit vector field to be totaly geodesic. The Theorem 2 contains a necessary and sufficient condition on strongly normal unit vector field to be minimal. In Section 4 we we apply the Lemma 2 to the case of a unit sphere (Lemma 4) and describe the geodesic unit vector fields on the sphere with totally geodesic property (Theorem 5). We also introduce a notion of covariantly normal unit vector field and prove that within this class the Hopf vector field is a unique one with a totally geodesic property (Theorem 3). This theorem is a revised and simplified version of Theorem 2.1 from [27]. The Section 5 contains an observation that the Hopf vector field on a unit sphere provides an example of global imbedding of Sasakian space form into Sasakian manifold as a Sasakian space form (Theorem 6).

2. Some Preliminaries

2.1. Sasaki Metric

Let (M,g) be n-dimensional Riemannian manifold with metric g. Denote by $\langle\cdot,\cdot\rangle$ a scalar product with respect to g. A natural Riemannian metric on the tangent bundle has been defined by S. Sasaki [20]. We describe it briefly in terms of the *connection map*.

At each point $Q=(q,\xi)\in TM$ the tangent space T_QTM can be split into the so-called *vertical* and *horizontal* parts:

$$T_QTM=\mathcal{H}_QTM\oplus\mathcal{V}_QTM.$$

The vertical part $\mathcal{V}_QTM$ is tangent to the fiber, while the horizontal part is transversal to it. If $(u^1,\dots,u^n;\xi^1,\dots,\xi^n)$ form the natural induced local coordinate system on TM, then for $\tilde X\in T_QTM^n$ we have

$$\tilde X=\tilde X^i\partial/\partial u^i+\tilde X^{n+i}\partial/\partial\xi^i$$

with respect to the natural frame $\{\partial/\partial u^i,\partial/\partial\xi^i\}$ on TM.

Denote by $\pi:TM\to M$ the tangent bundle projection map. Then its differential $\pi_*:T_QTM\to T_qM$ acts on $\tilde X$ as $\pi_*\tilde X=\tilde X^i\partial/\partial x^i$ and defines a linear isomorphism between $\mathcal{V}_QTM$ and T_qM.

The so-called *connection map* $K:T_QTM\to T_qM$ acts on $\tilde X$ by the rule $K\tilde X=(\tilde X^{n+i}+\Gamma^i_{jk}\xi^j\tilde X^k)\partial/\partial u^i$ and defines a linear isomorphism between $\mathcal{H}_QTM$ and T_qM. The images $\pi_*\tilde X$ and $K\tilde X$ are called *horizontal* and *vertical* projections of $\tilde X$, respectively. It is easy to see that $\mathcal{V}_Q=\ker\pi_*|_Q,\ \mathcal{H}_Q=\ker K|_Q$.

Let $\tilde X,\tilde Y\in T_QTM$. The *Sasaki metric* on TM is defined by the following scalar product

$$\langle\langle\tilde X,\tilde Y\rangle\rangle\big|_Q=\langle\pi_*\tilde X,\pi_*\tilde Y\rangle\big|_q+\langle K\tilde X,K\tilde Y\rangle\big|_q$$

at each point $Q=(q,\xi)$. Horizontal and vertical subspaces are mutually orthogonal with respect to Sasaki metric.

The operations inverse to projections are called *lifts*. Namely, if $X\in T_qM^n$, then $X^h=X^i\partial/\partial u^i-\Gamma^i_{jk}\xi^jX^k\partial/\partial\xi^i$ is in $\mathcal{H}_QTM$ and is called the *horizontal lift* of X, and $X^v=X^i\partial/\partial\xi^i$ is in $\mathcal{V}_QTM$ and is called the *vertical lift* of X.

The Sasaki metric can be completely defined by scalar product of combinations of lifts of vector fields from M to TM as

$$\langle\langle X^h,Y^h\rangle\rangle\big|_Q=\langle X,Y\rangle\big|_q,\ \ \langle\langle X^h,Y^v\rangle\rangle\big|_Q=0,\ \ \langle\langle X^v,Y^v\rangle\rangle\big|_Q=\langle X,Y\rangle\big|_q.$$

2.2. Harmonic and Minimal Unit Vector Fields

Suppose, as above, that $u := (u^1, \dots, u^n)$ are the local coordinates on M^n. Denote by $(u, \xi) := (u^1, \dots, u^n; \xi^1, \dots, \xi^n)$ the natural local coordinates in the tangent bundle TM^n. If $\xi(u)$ is a (unit) vector field on M^n, then it defines a mapping

$$\xi : M^n \to TM^n \quad \text{or} \quad \xi : M^n \to T_1M^n, \quad \text{if } |\xi| = 1,$$

given by $\xi(u) = (u, \xi(u))$.

For the mappings $f : (M, g) \to (N, h)$ between Riemannian manifolds the *energy* of f is defined as

$$E(f) := \frac{1}{2}\int_M |d\,f|^2\, dVol_M,$$

where $|d\,f|$ is a norm of 1-form $d\,f$ in the co-tangent bundle T^*M. Supposing on T_1M the Sasaki metric, the following definition becomes natural.

Definition 1. A unit vector field is called *harmonic*, if it is a critical point of energy functional of mapping $\xi : M^n \to T_1M^n$.

Up to an additive constant, the energy functional of the mapping the is a total bending of a unit vector field [24]

$$B(\xi) := c_n \int_M |\nabla\xi\,|^2\, dVol_M,$$

where c_n is some normalizing constant and $|\nabla\xi|^2 = \sum_{i=1}^n |\nabla_{e_i}\xi|^2$ with respect to orthonormal frame $e_1, \dots e_n$.

Introduce a point-wise linear operator $A_\xi : T_qM^n \to \xi_q^\perp$, acting as

$$A_\xi X = -\nabla_X\xi.$$

In case of integrable distribution $\xi^\perp$, the unit vector field ξ is called *holonomic* [1]. In this case the operator A_ξ is symmetric and is known as Weingarten or a *shape operator* for each hypersurface of the foliation. In general, A_ξ is not symmetric, but formally preserves the Codazzi equation. Namely, a covariant derivative of A_ξ is defined by

$$-(\nabla_X A_\xi)Y = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi. \tag{1}$$

Then for the curvature operator of M^n we can write down the Codazzi-type equation

$$R(X, Y)\xi = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y.$$

From this viewpoint, it is natural to call the operator A_ξ as *non-holonomic shape* operator. Remark, that the right hand side is, up to constant, a *skew symmetric part* of covariant derivative of A_ξ.

Introduce a symmetric tensor field

$$Hess_\xi(X, Y) = \frac{1}{2}\big[(\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y\big], \tag{2}$$

which is a *symmetric part* of covariant derivative of A_ξ. The trace

$$-\sum_{i=1}^{n} Hess_\xi(e_i, e_i) := \Delta\xi,$$

where $e_1, \dots e_n$ is an orthonormal frame, is known as *rough Laplacian* [2] of the field ξ. Therefore, one can treat the tensor field (2) as a *rough Hessian* of the field.

With respect to given above notations, the unit vector field is harmonic if and only if [24]

$$\Delta\xi = -|\nabla\xi|^2\xi.$$

Introduce a tensor field

$$Hm_\xi(X,Y) = \frac{1}{2}\big[R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X\big], \tag{3}$$

which is a symmetric part of tensor field $R(\xi, A_\xi X)Y$. The trace

$$trace\, Hm_\xi := \sum_{i=1}^{n} Hm_\xi(e_i, e_i)$$

is responsible for harmonicity of mapping $\xi : M^n \to T_1M^n$ in terms of general notion of harmonic maps [10]. Precisely, a *harmonic* unit vector field ξ defines a *harmonic mapping* $\xi : M^n \to T_1M^n$ if and only if [11]

$$trace\, Hm_\xi = 0.$$

From this viewpoint, it is natural to call the tensor field (3) as *harmonicity tensor* of the field ξ.

Consider now the image $\xi(M^n) \subset T_1M^n$ with a pull-back Sasaki metric.

Definition 2. A unit vector field ξ on Riemannian manifold M^n is called minimal if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is minimal submanifold in the unit tangent bundle T_1M^n with Sasaki metric.

A number of results on minimal unit vector fields one can find in [4, 5, 6, 8, 12, 13, 14, 15, 16, 17, 19, 21, 22, 23]. In [25], the author has found explicitly the second fundamental form of $\xi(M^n)$ and presented some examples of unit vector fields of *constant mean curvature*.

3. Totally Geodesic Unit Vector Fields

Definition 3. A unit vector field ξ on Riemannian manifold M^n is called totally geodesic if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is totally geodesic submanifold in the unit tangent bundle T_1M^n with Sasaki metric.

Using the explicit expression for the second fundamental form [25], the author gave a full description of the totally geodesic (local) unit vector fields on 2-dimensional Riemannian manifold.

Theorem 1. [28] *Let (M^2, g) be a Riemannian manifold with a sign-preserving Gaussian curvature K. Then M admits a totally geodesic unit vector field ξ if and only if there is a local parametrization of M with respect to which the metric g is of the form*

$$ds^2 = du^2 + \sin^2\alpha(u)\, dv^2,$$

where $\alpha(u)$ solves the differential equation $\dfrac{d\alpha}{du} = 1 - \dfrac{a+1}{\cos\alpha}$*. The corresponding local unit vector field ξ is of the form*

$$\xi = \cos(av + \omega_0)\,\partial_u + \frac{\sin(av+\omega_0)}{\sin\alpha(u)}\,\partial_v,$$

where $a, \omega_0 = const$.

For the case of *flat* Riemannian 2-manifold, the totally geodesic unit vector field is either parallel or moves helically along a pencil of parallel straight lines on a plane with a constant angle speed [26] . It is easy to see that the following corollary is true.

Corollary 1. *Integral trajectories of a totally geodesic (local) unit vector field on the non-flat Riemannian manifold M^2 are locally conformally equivalent to the integral trajectories of totally geodesic unit vector field on a plane. Moreover, with respect to Cartesian coordinates (x, y) on the plane, these integral trajectories are*

$$\begin{aligned} &x = c && \text{for } a=0,\\ &y(x) = -\tfrac{1}{a}\ln|\sin(ax)| + c && \text{for } a \neq 0, \end{aligned}$$

where c is a parameter.

In what follows, we present a new differential equation with respect to a unit vector field such that its solution generates a totally geodesic submanifold in T_1M^n.

In terms of horizontal and vertical lifts of vector fields from the base to its tangent bundle, the differential of mapping $\xi : M^n \to TM^n$ is acting as

$$\xi_* X = X^h + (\nabla_X \xi)^v = X^h - (A_\xi X)^v, \tag{4}$$

where ∇ means Levi-Civita connection on M^n and the lifts are considered to points of $\xi(M^n)$.

It is well known that if ξ is a *unit* vector field on M^n, then the vertical lift ξ^v is a *unit normal* vector field on a hypersurface $T_1M^n \subset TM^n$. Since ξ is of unit length, $\xi_* X \perp \xi^v$ and hence in this case $\xi_* : TM^n \to T(T_1M^n)$.

Denote by $A^t_\xi : \xi^\perp_q \to T_qM^n$ a formal adjoint operator

$$\langle A_\xi X, Y\rangle_q = \langle X, A^t_\xi Y\rangle_q.$$

Denote by $\xi^\perp$ a distribution on M^n with ξ as its normal unit vector field. Then for each vector field $N \in \xi^\perp$, the vector field

$$\tilde N = (A^t_\xi N)^h + N^v \tag{5}$$

is normal to $\xi(M^n)$. Thus, (5) presents the normal distribution on $\xi(M^n)$.

Lemma 1. *Let M^n be Riemannian manifold and T_1M^n its unit tangent bundle with Sasaki metric. Let ξ a smooth (local) unit vector field on M^n. The second fundamental form $\tilde\Omega_{\tilde N}$ of $\xi(M^n) \subset T_1M^n$ with respect to the normal vector field* (5) *is of the form*

$$\tilde\Omega_{\tilde N}(\xi_*X, \xi_*Y) = -\langle Hess_\xi(X,Y) + A_\xi Hm_\xi(X,Y), N\rangle, \tag{6}$$

where X and Y are arbitrary vector fields on M^n.

Proof: By definition, we have

$$\tilde\Omega_{\tilde N}(\xi_*X, \xi_*Y) = \langle\langle \tilde\nabla_{\xi_*X}\, \xi_*Y, \tilde N\rangle\rangle_{(q,\xi(q))},$$

where $\tilde\nabla$ is the Levi-Civita connection of Sasaki metric on TM^n. To calculate $\tilde\nabla_{\xi_*X}\, \xi_*Y$, we can use the formulas [18]

$$\begin{aligned}
&\tilde\nabla_{X^h} Y^h = (\nabla_X Y)^h - \tfrac12 (R(X,Y)\xi)^v, \quad \tilde\nabla_{X^v} Y^h = \tfrac12 (R(\xi,X)Y)^h,\\
&\tilde\nabla_{X^h} Y^v = (\nabla_X Y)^v \; + \tfrac12 (R(\xi,Y)X)^h, \;\; \tilde\nabla_{X^v} Y^v = 0.
\end{aligned}$$

A direct calculation yields

$$\tilde\nabla_{\xi_*X}\, \xi_*Y = \Big(\nabla_X Y + \tfrac12 R(\xi, \nabla_X\xi)Y + \tfrac12 R(\xi,\nabla_Y\xi)X\Big)^h + \Big(\nabla_X\nabla_Y\xi - \frac12 R(X,Y)\xi\Big)^v.$$

The derivative above is not tangent to $\xi(M^n)$. It contains a projection on "external" normal vector field, i.e. on ξ^v which is a unit normal of T_1M^n inside TM^n. To correct the situation, we should subtract this projection, namely $-\langle\nabla_X\xi, \nabla_Y\xi\rangle\xi$, from the vertical part of the derivative.

Therefore, we have

$$\tilde\Omega_{\tilde N}(\xi_*X, \xi_*Y) = \langle \nabla_X\nabla_Y\xi + \langle\nabla_X\xi, \nabla_Y\xi\rangle\xi - \frac12 R(X,Y)\xi, N\rangle + \langle \nabla_X Y + \frac12 R(\xi,\nabla_X\xi)Y + \frac12 R(\xi,\nabla_Y\xi)X, A^t_\xi N\rangle$$

or, equivalently,

$$\tilde\Omega_{\tilde N}(\xi_*X,\xi_*Y)=\langle\nabla_X\nabla_Y\xi+\langle\nabla_X\xi,\nabla_Y\xi\rangle\xi-\frac{1}{2}R(X,Y)\xi+ \\ A_\xi(\nabla_XY+\frac{1}{2}R(\xi,\nabla_X\xi)Y+\frac{1}{2}R(\xi,\nabla_Y\xi)X),N\rangle.$$

Taking into account (1), (2), (3) and (5), and also

$$R(X,Y)\xi=\nabla_X\nabla_Y\xi-\nabla_Y\nabla_X\xi-\nabla_{[X,Y]}\xi,$$

we can write

$$\tilde\Omega_{\tilde N}(\xi_*X,\xi_*Y)=-\langle Hess_\xi(X,Y)+A_\xi Hm_\xi(X,Y),N\rangle$$

which completes the proof. □

Lemma 2. *Let M^n be Riemannian manifold and T_1M^n its unit tangent bundle with Sasaki metric. Let ξ be a smooth (local) unit vector field on M^n. The vector field ξ generates a totally geodesic submanifold $\xi(M^n)\subset T_1M^n$ if and only if ξ satisfies*

$$Hess_\xi(X,Y)+A_\xi Hm_\xi(X,Y)-\langle A_\xi X,A_\xi Y\rangle\,\xi=0 \tag{7}$$

for all (local) vector fields X,Y on M^n.

Proof: Taking into account (6), the condition on ξ to be totally geodesic takes the form

$$-Hess_\xi(X,Y)-A_\xi Hm_\xi(X,Y)=\lambda\,\xi.$$

Multiplying the equation above by ξ, we can find easily $\lambda=-\langle A_\xi X,A_\xi Y\rangle$. □

Follow [16], we call a unit vector field ξ *strongly normal* if

$$\langle(\nabla_XA_\xi)Y,Z\rangle=0$$

for all $X,Y,Z\in\xi^\perp$. In other words, $(\nabla_XA_\xi)Y=\lambda\,\xi$ for all $X,Y\in\xi^\perp$. It is easy to find the function λ. Indeed, we have

$$\lambda=\langle(\nabla_XA_\xi)Y,\xi\rangle=\langle\nabla_{\nabla_XY}\xi-\nabla_X\nabla_Y\xi,\xi\rangle= \\ -\langle\nabla_X\nabla_Y\xi,\xi\rangle=\langle\nabla_X\xi,\nabla_Y\xi\rangle.$$

Thus, the strongly normal unit vector field can be characterized by the equation

$$(\nabla_XA_\xi)Y=\langle A_\xi X,A_\xi Y\rangle\,\xi \tag{8}$$

for all $X,Y\in\xi^\perp$.

The strong normality condition highly simplifies the second fundamental form of $\xi(M^n)\subset T_1M^n$. An orthonormal frame $e_1,e_2,\dots,e_n$ is called *adapted* to the field ξ if $e_1=\xi$ and $e_2,\dots,e_n\in\xi^\perp$.

Lemma 3. *Let ξ be a unit strongly normal vector field on Riemannian manifold M^n. With respect to the adapted frame, the matrical components of the second fundamental form of $\xi(M^n) \subset T_1(M^n)$ simultaneously take the form*

$$\tilde\Omega_{\tilde N} = \begin{pmatrix} * & * & \dots & * \\ * & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \dots & 0 \end{pmatrix}.$$

Proof: Set $N_\sigma = e_\sigma \quad (\sigma = 2, \dots, n)$. The condition (8) implies

$$R(X,Y)\xi = 0, \quad Hess_\xi(X,Y) = \langle A_\xi X, A_\xi Y\rangle \xi, \quad Hm_\xi(X,Y) \sim \xi$$

for all $X, Y \in \xi^\perp$. Therefore, with respect to the adapted frame

$$\tilde\Omega_\sigma(\xi_* e_\alpha, \xi_* e_\beta) = 0 \quad (\alpha, \beta = 2, \dots, n)$$

for all $\sigma = 2, \dots, n$. □

The following assertion is a natural corollary of the Lemma 3 .

Theorem 2. *Let ξ be a unit strongly normal vector field. Denote by k the geodesic curvature of its integral trajectories and by ν the principal normal unit vector field of the trajectories. The field ξ is minimal if and only if*

$$k[\xi,\nu] + \xi(k)\nu - kA_\xi R(\nu,\xi)\xi + k^2\xi = 0$$

where $[\xi,\nu] = \nabla_\xi \nu - \nabla_\nu \xi$.

Proof: Indeed,

$$\tilde\Omega_\sigma(\xi_* e_1, \xi_* e_1) = -\langle Hess_\xi(\xi,\xi) + A_\xi Hm_\xi(\xi,\xi), e_\sigma\rangle$$

Denote by ν a vector field of the principal normals of ξ-integral trajectories and by k their geodesic curvature function. Then

$$Hess_\xi(\xi,\xi) = \nabla_{\nabla_\xi \xi} - \nabla_\xi \nabla_\xi \xi = k\nabla_\nu \xi - \nabla_\xi(k\nu) = k[\nu,\xi] - \xi(k)\nu,$$

$$Hm_\xi(\xi,\xi) = -R(\xi, \nabla_\xi \xi)\xi = -kR(\xi,\nu)\xi$$

and we get

$$\tilde\Omega_\sigma(\xi_* e_1, \xi_* e_1) = \langle k[\xi,\nu] + \xi(k)\nu - kA_\xi R(\nu,\xi)\xi, e_\sigma\rangle.$$

Finally, to be minimal, the field ξ should satisfy

$$k[\xi,\nu] + \xi(k)\nu - kA_\xi R(\nu,\xi)\xi = \lambda\,\xi.$$

Multiplying by ξ, we get

$$\lambda = k\langle [\xi,\nu], \xi\rangle = k\langle \nabla_\xi \nu, \xi\rangle = -k^2,$$

which completes the proof. □

Thus, we get the following.

Corollary 2. [16] *Every unit strongly normal geodesic vector field is minimal.*

Most of examples of minimal unit vector fields in [16] are based on this Corollary.

4. The Case of a Unit Sphere

If the manifold is a unit sphere S^{n+1}, the equation (7) can be essentially simplified.

Lemma 4. *A unit (local) vector field ξ on a unit sphere S^{n+1} generates a totally geodesic submanifold $\xi(S^{n+1}) \subset T_1S^{n+1}$ if and only if ξ satisfies*

$$(\nabla_X A_\xi)Y = \frac{1}{2}\Big[(\mathcal{L}_\xi\, g)(X,Y)\, A_\xi\xi + \langle \xi, X\rangle\,(A_\xi^2 Y + Y) + \langle \xi, Y\rangle\,(A_\xi^2 X - X)\Big] + \langle A_\xi X, A_\xi Y\rangle\,\xi, \tag{9}$$

where $(\mathcal{L}_\xi\, g)(X,Y) = \langle \nabla_X\xi, Y\rangle + \langle X, \nabla_Y\xi\rangle$ is a Lie derivative of metric tensor in a direction of ξ.

Proof: Indeed, on a unit sphere

$$(\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y = R(X,Y)\xi = \langle \xi, Y\rangle X - \langle \xi, X\rangle Y.$$

Hence,

$$Hess_\xi(X,Y) = (\nabla_X A_\xi)Y + \frac{1}{2}[\langle \xi, Y\rangle X - \langle \xi, X\rangle Y].$$

For $Hm_\xi(X,Y)$ we have

$$Hm_\xi(X,Y) = \frac{1}{2}\Big[\langle \nabla_X\xi, Y\rangle\xi - \langle \xi, Y\rangle\nabla_X\xi + \langle \nabla_Y\xi, X\rangle\xi - \langle \xi, X\rangle\nabla_Y\xi\Big] = \frac{1}{2}(\mathcal{L}_\xi\, g)(X,Y)\,\xi + \frac{1}{2}\Big[\langle \xi, Y\rangle A_\xi X + \langle \xi, X\rangle A_\xi Y\Big].$$

Finally, we find

$$(\nabla_X A_\xi)Y = \frac{1}{2}\Big[(\mathcal{L}_\xi\, g)(X,Y)\, A_\xi\xi + \langle \xi, X\rangle\,(A_\xi^2 Y + Y) + \langle \xi, Y\rangle\,(A_\xi^2 X - X)\Big] + \langle A_\xi X, A_\xi Y\rangle\,\xi.$$

□

Remind that the operator A_ξ is symmetric if and only if the field ξ is holonomic, and is skew-symmetric if and only if the field ξ is a Killing vector field. Both types of these fields can be included into a class of *covariantly normal* unit vector fields.

Definition 4. A regular unit vector field on Riemannian manifold is said to be covariantly normal if the operator $A_\xi : TM \to \xi^\perp$ defined by $A_\xi X = -\nabla_X \xi$ satisfies the normality condition

$$A_\xi^t A_\xi = A_\xi A_\xi^t$$

with respect to some orthonormal frame.

The integral trajectories of holonomic and Killing unit vector fields are always geodesic. Every covariantly normal unit vector field possesses this property.

Lemma 5. *Integral trajectories of a covariantly normal unit vector field are geodesic lines.*

Proof: Suppose ξ is a unit covariantly normal vector field on a Riemannian manifold M^{n+1}. Find a unit vector field ν_1 such that

$$\nabla_\xi \xi = -k\nu_1.$$

Geometrically, the function k is a geodesic curvature of the integral trajectory of the field ξ.

Complete up the pair (ξ, ν_1) to the orthonormal frame $(\xi, \nu_1, \dots \nu_n)$. Then we can set

$$\nabla_\xi \xi = -k\nu_1, \quad \nabla_{\nu_\alpha} \xi = -a_\alpha^\beta \nu_\beta,$$

where $\alpha, \beta = 1, \dots, n$. With respect to the frame $(\xi, \nu_1, \dots \nu_n)$ the matrix A_ξ takes the form

$$-A_\xi = \begin{pmatrix} 0 & k & 0 & \dots & 0 \\ 0 & a_1^1 & a_2^1 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1^n & a_2^n & \dots & a_n^n \end{pmatrix}$$

and as a consequence

$$-A_\xi^t = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ k & a_1^1 & a_1^2 & \dots & a_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_n^1 & a_n^2 & \dots & a_n^n \end{pmatrix}.$$

Therefore,

$$A_\xi A_\xi^t = \begin{pmatrix} k^2 & ka_1^1 & \dots & ka_1^n \\ ka_1^1 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ ka_1^n & * & \dots & * \end{pmatrix}, \quad A_\xi^t A_\xi = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

and we conclude $k = 0$.

□

Now we can easily prove the following theorem.

Theorem 3. *Let ξ be a global covariantly normal unit vector field on a unit sphere S^{n+1}. Then ξ is a totally geodesic if and only if $n = 2m$ and ξ is a Hopf vector field.*

Proof: Suppose ξ is covariantly normal and totally geodesic. Then

$$A_\xi \xi = -\nabla_\xi \xi = 0$$

by Lemma 5 and the equation (9) takes the form

$$(\nabla_X A_\xi)Y = \frac{1}{2}\Big[\langle \xi, X\rangle\,(A_\xi^2 Y + Y) + \langle \xi, Y\rangle\,(A_\xi^2 X - X)\Big] + \langle A_\xi X, A_\xi Y\rangle\,\xi. \quad (10)$$

Setting $X = Y = \xi$ we get an identity. Set $Y = \xi$ and take arbitrary unit $X \perp \xi$. Then we get

$$2(\nabla_X A_\xi)\xi + X = A_\xi^2 X.$$

On the other hand, directly

$$(\nabla_X A_\xi)\xi = -(\nabla_X \nabla_\xi \xi - \nabla_{\nabla_X \xi}\xi) = A_\xi^2 X.$$

Hence,

$$A_\xi^2\big|_{\xi^\perp} = -E.$$

Therefore, $n = 2m$. Since A_ξ is real normal linear operator, there exists an orthonormal frame such that

$$A_\xi = \begin{pmatrix} 0 & & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

with zero all other entries. Therefore, $A_\xi + A_\xi^t = 0$ and ξ is a Killing vector field. Since ξ is supposed global, ξ is a Hopf vector field.

Finally, if we take $X, Y \perp \xi$, we get the equation

$$(\nabla_X A_\xi)Y = \langle A_\xi X, A_\xi Y\rangle \xi.$$

But for a Killing vector field ξ we have [16]

$$(\nabla_X A_\xi)Y = R(\xi, X)Y = \langle X, Y\rangle \xi.$$

Since ξ is a Hopf vector field, $\langle A_\xi X, A_\xi Y\rangle = \langle X, Y\rangle$. So, in this case we have an identity.

If we suppose now that ξ is a Hopf vector field on a unit sphere, then ξ is covariantly normal as a Killing vector field and totally geodesic [27] as a characteristic vector field of a standard contact metric structure on S^{2m+1}. □

Theorem 3 is a correct and simplified version of Theorem 2.1 [27], where the normality of the operator A_ξ was implicitly used in a proof.

In the case of a *weaker condition* on the field ξ to be only a *geodesic* one, the result is not so definite. We begin with some preparations.

The almost complex structure on TM^n is defined by

$$JX^h = X^v, \quad JX^v = -X^h$$

for all vector field X on M^n. Thus, TM^n with Sasaki metric is an almost Kählerian manifold. It is Kählerian if and only if M^n is flat [9].

The unit tangent bundle T_1M^n is a hypersurface in TM^n with a unit normal vector ξ^v at each point $(q,\xi) \in T_1M^n$. Define a unit vector field $\bar\xi$, a 1-form $\bar\eta$ and a $(1,1)$ tensor field $\bar\varphi$ on T_1M^n by

$$\bar\xi = -J\xi^v = \xi^h, \quad JX = \bar\varphi X + \bar\eta(X)\xi^v.$$

The triple $(\bar\xi, \bar\eta, \bar\varphi)$ form a standard almost contact structure on T_1M^n with Sasaki metric g_S. This structure is not almost contact *metric* one. By taking

$$\tilde\xi = 2\bar\xi = 2\xi^h, \quad \tilde\eta = \frac{1}{2}\bar\eta, \quad \tilde\varphi = \bar\varphi, \quad g_{cm} = \frac{1}{4}g_S$$

at each point $(q,\xi) \in T_1M^n$, we get the *almost contact metric structure* $(\tilde\xi, \tilde\eta, \tilde\varphi)$ on (T_1M^n, g_{cm}).

In a case of a general almost contact metric manifold $(\tilde M, \tilde\xi, \tilde\eta, \tilde\varphi, \tilde g)$ the following definition is known [7].

Definition 5. A submanifold N of a contact metric manifold $(\tilde M, \tilde\xi, \tilde\eta, \tilde\varphi, \tilde g)$ is called invariant if $\tilde\varphi(T_pN) \subset T_pN$ and anti-invariant if $\tilde\varphi(T_pN) \subset (T_pN)^\perp$ for every $p \in N$.

If N is the invariant submanifold, then the characteristic vector field $\tilde\xi$ is *tangent* to N at each of its points.

After all mentioned above, the following definition is natural [3].

Definition 6. A unit vector field ξ on a Riemannian manifold (M^n, g) is called invariant (anti-invariant) is the submanifold $\xi(M^n) \subset (T_1M^n, g_{cm})$ is invariant (anti-invariant).

It is easy to see from (4) that the *invariant* unit vector field is always a geodesic one, i.e. its integral trajectories are geodesic lines.

Binh T.Q., Boeckx E. and Vanhecke L. have considered this kind of unit vector fields [3] and proved the following theorem.

Theorem 4. *A unit vector field ξ on (M^n, g) is invariant if and only if $(\tilde\xi = \xi\,,\ \tilde\eta = \langle\cdot,\xi\rangle_g\,,\ \tilde\varphi = A_\xi)$ is an almost contact structure on M^n. In particular, ξ is a geodesic vector field on M^n and $n = 2m+1$.*

Now we can formulate the result.

Theorem 5. *A unit geodesic vector field ξ on S^{n+1} is totally geodesic if and only if $n = 2m$ and ξ is a strongly normal invariant unit vector field.*

Proof: Suppose ξ is a geodesic and totally geodesic unit vector field. Then $A_\xi \xi = 0$ and the equation (9) takes the form (10). Follow the proof of the Theorem 3, we come to the following conditions on the field ξ:

$$A_\xi^2 X = -X, \quad (\nabla_X A_\xi)Y = \langle A_\xi X, A_\xi Y \rangle \, \xi \tag{11}$$

for all $X, Y \in \xi^\perp$. From $(11)_1$ we conclude that $n = 2m$. Comparing $(11)_2$ with (8), we see that ξ is a strongly normal vector field.
Consider now a $(1,1)$ tensor field $\varphi = A_\xi = -\nabla\xi$ and a 1-form $\eta = \langle \,\cdot\,, \xi \,\rangle$. Taking into account $(11)_1$ and $A_\xi \xi = 0$, we see that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(X) = 1$$

for any vector field X on the sphere. Therefore, the triple

$$\tilde{\varphi} = A\xi, \quad \tilde{\xi} = \xi, \quad \tilde{\eta} = \langle \,\cdot\,, \xi \,\rangle$$

form an *almost contact structure* with the field ξ as a characteristic vector field of this structure. By Theorem 4, the field ξ is invariant.
Conversely, suppose ξ is strongly normal and invariant on S^{n+1}. Then, by Theorem 4, ξ is geodesic and $n = 2m$. The rest of the proof is a direct checking of the formula (10). □

5. A Remarkable Property of the Hopf Vector Field

It is well-known that for a unit sphere S^n the standard contact metric structure on $T_1 S^n$ is a Sasakian one. If ξ is a Hopf unit vector field on S^{2m+1}, then ξ is a characteristic vector field of a standard contact metric structure on the unit sphere S^{2m+1}. By Theorem 4, the submanifold $\xi(S^{2m+1})$ is invariant submanifold in $T_1 S^{2m+1}$. Therefore, $\xi(S^{2m+1})$ *is also Sasakian* with respect to the induced structure [29]. Since the Hopf vector field is strongly normal, by Theorem 5, the submanifold $\xi(S^{2m+1})$ is totally geodesic. The sectional curvature of the submanifold $\xi(S^{2m+1})$ was found in [27] and implies a remarkable corollary.

Theorem 6. *Let ξ be a Hopf vector field on the unit sphere S^{2m+1}. With respect to the induced structure, the manifold $\xi(S^{2m+1})$ is a Sasakian space form of φ-curvature 5/4.*

In other words, the Hopf vector field provides an example of embedding of a *Sasakian space form* of φ-curvature 1 into *Sasakian manifold* such that the image is contact, totally geodesic *Sasakian space form* of φ-curvature 5/4 with respect to the induced structure.

Acknowledgements

The author thanks the Organizing Committee of the Conference for a partial financial support and the Euro-Commission for a travel grant. A special gratitude to professor Ivaïlo Mladenov for invitation and hospitality.

References

[1] Aminov Yu., *The Geometry of Vector Fields,* Gordon and Breach Publ., 2000.

[2] Besse A., *Einstein Manifolds,* Springer-Verlag, 1987.

[3] Binh T.Q., Boeckx E. and Vanhecke L., *Invariant and Anti-invariant Unit Vector Fields,* to appear

[4] Boeckx E. and Vanhecke L., *Harmonic and Minimal Vector Fields on Tangent and Unit Tangent Bundles,* Differential Geom. Appl.**13** (2000) 77–93.

[5] Boeckx E. end Vanhecke L., *Harmonic and Minimal Radial Vector Fields,* Acta Math. Hungar. **90** (2001) 317–331.

[6] Boeckx E. and Vanhecke L., *Isoparametric Functions and Harmonic and Minimal Unit Vector Fields,* Contemporary Math. **288** (2001) 20–31.

[7] Blair D., *Riemannian Geometry of Contact and Symplectic Manifolds,* Progress in Mathematics **203**, Birkhäuser, 2001.

[8] Brito F., Chacon P. and Naveira A., *On the Volume of Vector Fields on Spaces of Constant Sectional Curvature,* Comment. Math. Helv. **79** (2004)300–316.

[9] Dombrowski P., *On the Geometry of Tangent Bundle,* J. Reine Angew. Math. **210/1-2** (1962) 73–88.

[10] Eels J. and Lemaire L., *A Report on Harmonic Maps,* Bull. London Math. Soc. **20** (1978) 385–524.

[11] Gil-Medrano O., *Relationship between Volume and Energy of Unit Vector Fields,* Diff. Geom. Appl., **15** (2001) 137–152.

[12] Gil-Medrano O. and Llinares-Fuster E., *Second Variation of Volume and Energy of Vector Fields. Stability of Hopf Vector Fields,* Math. Ann. **320** (2001) 531–545.

[13] Gil-Medrano O., González-Dávila J.C. and Vanhecke L., *Harmonic and Minimal Invariant Unit Vector Fields on Homogeneous Riemannian Manifolds,* Houston J. Math. **27** (2001) 377–409.

[14] Gil-Medrano O. and Llinares-Fuster E., *Minimal Unit Vector Fields,* Tôhoku Math. J. **54** (2002) 71 – 84.

[15] Gluck H. and Ziller W., *On the Volume of a Unit Vector Field on the Three-Sphere,* Comm. Math. Helv. **61** (1986) 177–192.

[16] González-Dávila J.C. and Vanhecke L., *Examples of Minimal Unit Vector Fields,* Ann Global Anal. Geom. **18** (2000), 385 – 404.

[17] González-Dávila J.C. and Vanhecke L., *Minimal and Harmonic Characteristic Vector Fields on Three-Dimensional Contact Mentic Manifolds,* Journal of Geometry **72** (2001) 65–76.

[18] Kowalski O., *Curvature of the Induced Riemannian Metric on the Tangent Bundle of a Riemannian Manifold,* J. Reine Angew. Math. **250** (1971) 124–129.

[19] Pedersen S., *Volumes of Vector Fields on Spheres,* Trans. Amer. Math. Soc. **336** (1993) 69-78.

[20] Sasaki S., *On the Differential Geometry of Tangent Bundles of Riemannian Manifolds,* Tôhoku Math. J. **10** (1958) 338–354.

[21] Tsukada K. and Vanhecke L., *Invariant Minimal Unit Vector Fields on Lie Groups,* Period. Math. Hungar. **40** (2000) 123–133.

[22] Tsukada K. and Vanhecke L., *Minimal and Harmonic Vector Fields on* $G_2(C^{m+2})$ *and Its Dual Space,* Monatsh. Math. **130** (2000) 143–154.

[23] Tsukada K. and Vanhecke L., *Minimality and Harmonicity for Hopf Vector Fields,* Illinois J. Math. **45** (2001) 441–451.

[24] Weigmink G., *Total Bending of Vector Fields on Riemannian Manifolds,* Math. Ann. **303** (1995) 325 – 344.

[25] Yampolsky A., *On the Mean Curvature of a Unit Vector Field,* Math. Publ. Debrecen, **60/1-2** (2002) 131 – 155.

[26] Yampolsky A., *On the Intrinsic Geometry of a Unit Vector Field,* Comment. Math. Univ. Carolinae **43/2** (2002) 299–317.

[27] Yampolsky A., *A Totally Geodesic Property of Hopf Vector Fields,* Acta Math. Hungar. **101/1-2** (2003) 73–92.

[28] Yampolsky A., *Full Description of Totally Geodesic Unit Vector Fields on 2-dimensional Riemannian Manifold,* Matematicheskaya Fizika, Analiz, Geometriya **11/3** (2004) 355–365.

[29] Yano K. and Kon M., *CR-submanifolds of Kählerian and Sasakian Manifolds,* Birkhäuser, 1983.