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Minimal and totally geodesic unit sections of the unit sphere bundles.

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We consider a real vector bundle $\mathcal{E}$ of rank p and a unit sphere bundle $\mathcal{E}_1 \subset \mathcal{E}$ over the Riemannian M^n with the Sasaki-type metric. A unit section of $\mathcal{E}_1$ gives rise to a submanifold in $\mathcal{E}_1$. We give some examples of local minimal unit sections and present a complete description of local totally geodesic unit sections of $\mathcal{E}_1$ in the simplest non-trivial case $p = 2$ and $n = 2$.
Keywords: Sasaki metric, unit sphere bundle, totally geodesic unit section.

Ямпольський О. Л., **Мінімальні і цілком геодезичні одиничні перерізи сферичних розшарувань.** Ми розглядаємо векторне розшарування $\mathcal{E}$ рангу p та одиничне розшарування $\mathcal{E}_1 \subset \mathcal{E}$ над рімановим многовидом M^n з метрикою Сасакі. Ми наводимо приклади мінімальних перерізів і надаємо повне вирішення задачі про цілком геодезичні перерізи $\mathcal{E}_1$ у найпростішому нетривіальному випадку, коли $p = 2$ і $n = 2$.
Ключові слова: метрика Сасакі, одиничне розшарування, цілком геодезичний одиничний переріз.

Ямпольский А. Л., **Минимальные и вполне геодезические сечения единичных сферических расслоений.** Мы рассматриваем вещественное векторное расслоение $\mathcal{E}$ ранга p и единичное расслоение $\mathcal{E}_1$ над римановым многообразием M^n с метрикой Сасаки. Мы приводим примеры локальных минимальных единичных сечений и даем полное решение задачи существования локальных вполне геодезических сечений $\mathcal{E}_1$ в простейшем нетривиальном случае, когда $p = 2$ и $n = 2$.
Ключевые слова: метрика Сасаки, сферическое расслоение, вполне геодезическое единичное сечение.

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Introduction

A. Borisenko [1] posed a problem on description of all totally geodesic submanifolds in a (unit) tangent bundle with Sasaki metric over a space of constant curvature.

A natural class of submanifolds in the (unit) tangent bundle is formed by (unit) vector fields on the base. H. Gluck and W. Ziller [2] posed a problem to find "the best organized" unit vector field on spheres and proposed on this role the field which gives rise to minimal submanifold in the unit tangent bundle with the Sasaki metric. Later on this meaning of unit vector field was extended the notion of *locally* minimal [3] and totally geodesic [4, 5] unit vector field. From this viewpoint the unit vector fields has been considered by many authors (see, e.g. [13, 14, 15, 16, 16, 17, 18]).

A natural generalization of tangent vector field is a section of given vector bundle $(\mathcal{E},\pi,\mathcal{B})$. If $\mathcal{B}$ is the Riemannian manifold and $\mathcal{E}$ is endowed with a fiberwise metric and compatible bundle connection, then one can define the Sasaki-type metric and consider a unit section as a harmonic map [6, 7] or as a locally minimal (totally geodesic) unit section of a subbundle $\mathcal{E}_1 \subset \mathcal{E}$ formed by the unit vectors of each fiber. The latter approach is presented in the given paper.

Section 1 contains necessary preparations. In Section 2 we study the simplest nontrivial case of vector bundle of rank 2 over 2 dimensional Riemannian manifold in details. We give a local description of the base manifold and the bundle connection for the case when the bundle admits a local unit totally geodesic section (Theorems 1 and 2). In Section 3 we give some some examples of minimal and totally geodesic sections of tangent and normal bundles.

The vector bundle, the base manifold and the sections are assumed smooth of class C^m $(m \geq 2)$ or analytic, if necessary.

1. The Sasaki-type metric on vector bundle

Let $(\mathcal{E},\pi,\mathcal{B})$ be a smooth real vector bundle of rank p over a smooth manifold B of dimension n. A *smooth section* is a smooth mapping $s:\mathcal{B}\to\mathcal{E}$ such that $\pi\circ s = id_{\mathcal{B}}$. By definition, $s(q)\in\mathcal{F}_q$, where $\mathcal{F}_q$ is a fiber over $q\in\mathcal{B}$ and the fiber $\mathcal{F}_q$ is a real p-dimensional vector space. The section could not exist globally but it is possible to find p linearly independent sections $s_1,\dots,s_p$ over a trivializing neighborhood $\mathcal{U}(u^1,\dots,u^n)\subset\mathcal{B}$. Then, for any $\xi\in T_u\mathcal{B}$, we have a decomposition $\xi=\xi^\alpha\, s_\alpha(u)$. The parameters $(u^1,\dots,u^n;\xi^1,\dots,\xi^p)$ form a *natural local coordinate system* in $\mathcal{U}\times\mathcal{F}\approx\mathcal{U}\times\mathbb{R}^p$. Any smooth local section $\xi:\mathcal{U}\to\mathcal{E}$ can be given by

$$\xi=\xi^\alpha(u)\, s_\alpha(u),$$

where $\xi^\alpha:\mathcal{U}\to\mathbb{R}$ some smooth functions. The image $\xi(\mathcal{U})\subset\mathcal{E}$ represents the analog of explicitly given submanifold with respect to natural local coordinate system. The *mail goal* of the paper is to study some geometrical properties of the locally given submanifold $\xi(\mathcal{U})$ in the case when $\mathcal{E}$ is endowed with the so-called *Sasaki-type* metric and the section ξ is of unit length.

Take a trivializing neighborhood $\mathcal{U} \subset \mathcal{B}$. Let $(u^1, \dots, u^n; \xi^1, \dots, \xi^p)$ be the natural local coordinate system on $\mathcal{E}$. A local frame

$$\tilde{\partial}_i := \frac{\partial}{\partial u^i}, \quad \tilde{\partial}_{n+\alpha} := \frac{\partial}{\partial \xi^\alpha}$$

is called *natural tangent local coordinate frame* over the restriction $\mathcal{E}|_\mathcal{U}$. Thus, for any local vector field $\tilde{X}$ on $\mathcal{E}|_\mathcal{U}$, we have a decomposition

$$\tilde{X} = \tilde{X}^i(u,\xi)\tilde{\partial}_i + \tilde{X}^{n+\alpha}(u,\xi)\tilde{\partial}_{n+\alpha}.$$

At each point $(u,\xi) \in T_{(u,\xi)}\mathcal{E}$ we have a decomposition $T_{(u,\xi)}\mathcal{E} = \mathcal{V}_{(u,\xi)}\mathcal{E} \oplus \mathcal{H}_{(u,\xi)}\mathcal{E}$, where $\mathcal{V}_{(u,\xi)}\mathcal{E}$ and $\mathcal{H}_{(u,\xi)}\mathcal{E}$ are tangent and transversal to the fiber at $u = \pi(u,\xi)$, respectively. The $\mathcal{V}_{(u,\xi)}\mathcal{E}$ is called *vertical* subspace and $\mathcal{H}_{(u,\xi)}\mathcal{E}$ is called *horizontal* subspace at $(u,\xi) \in \mathcal{E}$. The horizontal distribution $\mathcal{H}_{(u,\xi)}\mathcal{E}$ is called *bundle connection*. Over each trivializing neighborhood $\mathcal{U}$, the horizontal distribution can be defined by

$$\mathcal{H}|_\mathcal{U} = \bigcap_{\alpha=1}^{p} \ker(\theta^\alpha),$$

where $\theta^1, \dots, \theta^p$ is a collection of linearly independent smooth linear forms over $\pi^{-1}(\mathcal{U})$. The bundle connection is called *linear*, if the forms $\theta^1, \dots, \theta^p$ are taken by

$$\theta^\alpha = d\xi^\alpha + \gamma^\alpha_{\beta i}(u)\xi^\beta du^i.$$

The functions $\gamma^\alpha_{\beta i}$ are called *fiber bundle connection coefficients* and subject to the definite transformation law in a pass to the neighboring trivializing neighborhood (see [19] for details).

Denote by $\mathfrak{S}(\mathcal{B})$ and $\mathfrak{X}(\mathcal{B})$ the module of smooth sections of $\mathcal{E}$ and Lie algebra of smooth vector fields on $\mathcal{B}$, respectively. For any $\xi \in \mathfrak{S}(\mathcal{B})$ and any $X \in \mathfrak{X}(\mathcal{B})$, the section

$$\nabla^\mathcal{F}_X \xi := X^i \left(\frac{\partial \xi^\alpha}{\partial u^i} + \gamma^\alpha_{\beta i}\xi^\beta \right) s_\alpha$$

is called *fiber bundle covariant derivative* of the section ξ in a direction of the tangent vector field X.

The *connection map* $\mathcal{K} : T_{(u,\xi)}\mathcal{E} \to \mathcal{F}_u$ is defined locally by $\mathcal{K}\tilde{X} = (\tilde{X}^{n+\alpha} + \gamma^\alpha_{\beta i}\xi^\beta \tilde{X}^i)s_\alpha$. The *bundle projection differential* $\pi_* : T\mathcal{E} \to T\mathcal{B}$ acts by $\pi_*\tilde{X} = \tilde{X}^i\partial_i$, where $\partial_i = \pi_*(\tilde{\partial}_i) = \frac{\partial}{\partial u^i}$ are the vectors of the local coordinate frame over $\mathcal{U}$. These mappings possess the following easy-to-check properties

$$\ker \pi_* = \mathcal{V}_{(u,\xi)}, \quad \operatorname{im} \pi_* = T_u\mathcal{B},$$
$$\ker \mathcal{K} = \mathcal{H}_{(u,\xi)}, \quad \operatorname{im} K = \mathcal{F}_u$$

at each point $(u,\xi) \in \mathcal{E}$.

For any $X = X^i(u)\partial_i \in \mathfrak{X}(\mathcal{B})$, the vector field

$$X^h(u,\xi) = X^i(u)\tilde{\partial}_i - \gamma^\alpha_{\beta k}(u)X^k(u)\xi^\beta\,\tilde{\partial}_{n+\alpha}$$

is in $\mathcal{H}_{(u,\xi)}$ and is called *horizontal lift* of $X(u)$ to $T_{(u,\xi)}\mathcal{E}$. For any $\eta = \eta^\alpha(u)s_\alpha \in \mathfrak{S}(\mathcal{B})$, the vector field

$$\eta^v(u,\xi) = \eta^\alpha(u)\tilde{\partial}_{n+\alpha}$$

is in $\mathcal{V}_{(u,\xi)}$ and is called *vertical lift* of $\eta(u)$ to $T_{(u,\xi)}\mathcal{E}$.

For any smooth section $\xi : \mathcal{B} \to \mathcal{E}$, the section differential $\xi_* : T\mathcal{B} \to T\mathcal{E}$ acts by $\xi_* X = X^h + (\nabla^{\mathcal{F}}_X \xi)^v$ and hence $\mathcal{K}(\xi_* X) = \nabla^{\mathcal{F}}_X \xi$. A *fiber-wise metric* on $\mathcal{E}$ is a smooth function $g^{\mathcal{F}} : \mathcal{E} \to \mathbb{R}_+$ such that the restriction $g^{\mathcal{F}}|_{\mathcal{F}_u}$ is a positively definite quadratic form in $\xi^1, \dots, \xi^p$. A vector bundle is said to be *metrized* if it admits a fiber-wise metric. A fiber-wise metric is said to be *compatible with the bundle connection*, if

$$\partial_X(g^{\mathcal{F}}(\xi,\eta)) = g^{\mathcal{F}}(\nabla^{\mathcal{F}}_X\xi, \eta) + g^{\mathcal{F}}(\xi, \nabla^{\mathcal{F}}_X\eta).$$

From now on, suppose $\mathcal{B}$ is the Riemannian manifold $(\mathcal{B}, g^{\mathcal{B}})$ and $\mathcal{E}$ is a metrized vector bundle with the fiber-wise metric $g^{\mathcal{F}}$ compatible with the bundle connection.

Definition 1 *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a smooth vector bundle over the Riemannian manifold $(B, g^{\mathcal{B}})$ with a fiber-wise metric $g^{\mathcal{F}}$ compatible with the bundle connection $\nabla^{\mathcal{F}}$. Let $\tilde{X}, \tilde{Y}$ be smooth vector fields on $\mathcal{E}$. The Sasaki-type metric $g^{\mathcal{E}}$ on $\mathcal{E}$ is defined by the following scalar product*

$$g^{\mathcal{E}}(\tilde{X}, \tilde{Y}) = g^{\mathcal{B}}(\pi_*\tilde{X}, \pi_*\tilde{Y}) + g^{\mathcal{F}}(\mathcal{K}\tilde{X}, \mathcal{K}\tilde{Y}). \tag{1}$$

With respect to natural local coordinates on $\mathcal{E}$, the line element of $(\mathcal{E}, g^{\mathcal{E}})$ takes the form

$$d\tilde{s}^2_{\mathcal{E}} = ds^2_{\mathcal{B}} + |D^{\mathcal{F}}\xi|^2_{g^{\mathcal{F}}}, \tag{2}$$

where $ds^2_{\mathcal{B}}$ is the line element of the Riemannian base manifold $\mathcal{B}$ and $D^{\mathcal{F}}\xi = (d\xi^\alpha + \gamma^\alpha_{\beta i}\xi^\beta du^i)s_\alpha$ is the covariant differential of the "point" vector $\xi \in \mathcal{F}_u$ with respect to the bundle connection.

The horizontal and vertical subspaces are mutually orthogonal with respect to $g^{\mathcal{E}}$. In terms of lifts,

$$g^{\mathcal{E}}(X^h, Y^h) = g^{\mathcal{B}}(X, Y), \quad g^{\mathcal{E}}(X^h, \zeta^v) = 0, \quad g^{\mathcal{E}}(\eta^v, \zeta^v) = g^{\mathcal{F}}(\eta, \zeta).$$

A tri-linear mapping $R^{\mathcal{F}} : \mathfrak{X}(\mathcal{B}) \times \mathfrak{X}(\mathcal{B}) \times \mathfrak{S}(B) \to \mathfrak{S}(\mathcal{B})$ defined by

$$R^{\mathcal{F}}(X,Y)\xi = \nabla^{\mathcal{F}}_X\nabla^{\mathcal{F}}_Y\xi - \nabla^{\mathcal{F}}_Y\nabla^{\mathcal{F}}_X\xi - \nabla^{\mathcal{F}}_{[X,Y]}\xi$$

is called *curvature tensor of the bundle connection*. The bundle connection is said to be *flat* if $R^{\mathcal{F}}(X,Y)\xi = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{B}), \xi \in \mathfrak{S}(\mathcal{B})$.

Direct computations give the following formulas (c.f. [9]) .

Lemma 1 *Let $X, Y \in \mathfrak{X}(\mathcal{B})$ and $\eta, \zeta \in \mathfrak{S}(\mathcal{B})$. Then over each trivializing neighborhood the Lie-brackets of combinations of lifts to $T_{(u,\xi)}\mathcal{E}$ have the following expressions*

$$[X^h, Y^h] = [X, Y]^h - \big(R^{\mathcal{F}}(X, Y)\xi)\big)^v, \quad [X^h, \eta^v] = \big(\nabla^{\mathcal{F}}_X \eta\big)^v, \quad [\eta^v, \zeta^v] = 0.$$

By using the Koszul formula, it is easy find the Levi-Civita connection for the Sasaki-type metric on $\mathcal{E}$ (c.f. [10]).

Lemma 2 *Denote by $\tilde{\nabla}$ the Levi-Civita connection of $(\mathcal{E}, g^{\mathcal{E}})$. Let $X, Y \in \mathfrak{X}(\mathcal{B})$ and $\eta, \zeta \in \mathfrak{S}(\mathcal{B})$. Then over each trivializing neighborhood the covariant derivatives of combinations of lifts to $T_{(u,\xi)}\mathcal{E}$ have the following expressions*

$$\begin{aligned}
&\tilde{\nabla}_{X^h} Y^h = \big(\nabla^{\mathcal{B}}_X Y\big)^h - \big(\tfrac{1}{2} R^{\mathcal{F}}(X, Y)\xi\big)^v, \quad \tilde{\nabla}_{\eta^v} Y^h = \big(\tfrac{1}{2}\hat{R}^{\mathcal{F}}(\xi, \eta)Y\big)^h, \\
&\tilde{\nabla}_{X^h} \eta^v = \big(\nabla^{\mathcal{F}}_X \eta\big)^v + \big(\tfrac{1}{2}\hat{R}^{\mathcal{F}}(\xi, \eta)X\big)^h, \qquad \tilde{\nabla}_{\eta^v} \zeta^v = 0,
\end{aligned}$$

where $\hat{R}^{\mathcal{F}} : \mathfrak{S}(\mathcal{B}) \times \mathfrak{S}(\mathcal{B}) \times \mathfrak{X}(B) \to \mathfrak{X}(\mathcal{B})$ is defined by $g^{\mathcal{B}}(\hat{R}^{\mathcal{F}}(\xi, \eta)X, Y) = g^{\mathcal{F}}(R^{\mathcal{F}}(X, Y)\xi, \eta)$.

The tensor field $\hat{R}^{\mathcal{F}}$ is called *formally conjugate* to the bundle connection curvature tensor field $R^{\mathcal{F}}$. Lemma 2 implies the following remarks: the fibers of $\mathcal{E}$ are totally geodesic and flat submanifolds of $(\mathcal{E}, g^{\mathcal{E}})$; a single fiber normal bundle connection is defined by $\hat{R}^{\mathcal{F}}$; the horizontal distribution $\mathcal{H}$ is non-integrable (except the case of flat bundle connection) but totally geodesic one.

Lemmas 1 and 2 allows to calculate the curvature tenor of $(T\mathcal{E}, g^{\mathcal{E}})$.

Lemma 3 *Denote by $\tilde{R}$ a curvature tensor of $(\mathcal{E}, g^{\mathcal{E}})$. Then at each point $(u, \xi) \in \mathcal{E}$, the $\tilde{R}$ is completely defined by*

$$\begin{aligned}
&\tilde{R}(\eta^v, \zeta^v)\chi^v = 0, \\
&\tilde{R}(\eta^v, \zeta^v)Z^h = \Big(\hat{R}^{\mathcal{F}}(\eta, \zeta)Z + \tfrac{1}{4}\hat{R}^{\mathcal{F}}(\xi, \eta)\hat{R}^{\mathcal{F}}(\xi, \zeta)Z - \\
&\qquad \tfrac{1}{4}\hat{R}^{\mathcal{F}}(\xi, \zeta)\hat{R}^{\mathcal{F}}(\xi, \eta)Z\Big)^h, \\
&\tilde{R}(X^h, \zeta^v)\chi^v = -\Big(\tfrac{1}{2}\hat{R}^{\mathcal{F}}(\zeta, \chi)X + \tfrac{1}{4}\hat{R}^{\mathcal{F}}(\xi, \zeta)\hat{R}^{\mathcal{F}}(\xi, \chi)X\Big)^h, \\
&\tilde{R}(X^h, \zeta^v)Z^h = \Big(\tfrac{1}{2}R^{\mathcal{F}}(X, Z)\zeta + \tfrac{1}{4}R^{\mathcal{F}}(\hat{R}^{\mathcal{F}}(\xi, \zeta)Z, X)\xi\Big)^v + \\
&\qquad \Big(\tfrac{1}{2}(D_X \hat{R}^{\mathcal{F}})(\xi, \zeta)Z\Big)^h,
\end{aligned}$$

$$\begin{aligned}\tilde R(X^h,Y^h)\chi^v = & \Big(R^{\mathcal F}(X,Y)\chi+\tfrac14 R^{\mathcal F}(\hat R^{\mathcal F}(\xi,\chi)Y,X)\xi-\\ & \tfrac14 R^{\mathcal F}(\hat R^{\mathcal F}(\xi,\chi)X,Y)\xi\Big)^v+\\ & \tfrac12\Big((D_X\hat R^{\mathcal F})(\xi,\chi)Y-(D_Y\hat R^{\mathcal F})(\xi,\chi)X\Big)^h,\\ \tilde R(X^h,Y^h)Z^h = & \Big(R^{\mathcal B}(X,Y)Z+\tfrac14\hat R^{\mathcal F}(\xi,R^{\mathcal F}(X,Z)\xi)Y-\\ & \tfrac14\hat R^{\mathcal F}(\xi,R^{\mathcal F}(Y,Z)\xi)X+\tfrac12\hat R^{\mathcal F}(\xi,R^{\mathcal F}(X,Y)\xi)Z\Big)^h+\\ & \Big(\tfrac12(D_Z R^{\mathcal F})(X,Y)\xi\Big)^v,\end{aligned}$$

where $X,Y,Z\in\mathfrak X(\mathcal B)$, $\eta,\zeta,\chi\in\mathfrak S(\mathcal B)$, $R^{\mathcal B}$ is the Riemannian curvature tensor of $(\mathcal B,g^{\mathcal B})$ and

$$(D_X\hat R^{\mathcal F})(\xi,\eta)Z:=\nabla^{\mathcal B}_X\Big(\hat R^{\mathcal F}(\xi,\eta)Z\Big)-\hat R^{\mathcal F}(\xi,\nabla^{\mathcal F}_X\eta)Z-\hat R^{\mathcal F}(\xi,\eta)\nabla^{\mathcal B}_X Z,$$

$$(D_Z R^{\mathcal F})(X,Y)\xi:=\nabla^{\mathcal F}_Z\Big(R^{\mathcal F}(X,Y)\xi\Big)-R^{\mathcal F}(\nabla^{\mathcal B}_Z X,Y)\xi-R^{\mathcal F}(X,\nabla^{\mathcal B}_Z Y)\xi.$$

The proofs of Lemma 1, Lemma 2 and Lemma 3 are the step-by-step analogs of the proofs of the similar Lemmas for the normal bundle case [1].

Consider a single fiber $\mathcal F_u$. At each point $\xi\in\mathcal F_u$, we have $T_\xi(\mathcal F_{u_0})=Span(s_1^v,\dots,s_p^v)$ and $T^\perp_\xi(\mathcal F_u)=Span(\partial_1^h,\dots,\partial_n^h)$. By using the normal coordinates in a neighborhood of $u\in\mathcal B$ one can get $(\partial_1^h,\dots,\partial_n^h)$ as the orthonormal normal bundle frame along $\mathcal F_u$. The Ricci equation being applied to a single (totally geodesic) fiber yields the following corollary.

Corollary 1 *Let $(\mathcal E,g^{\mathcal E})$ be a vector bundle with the Sasaki-type metric. Denote by $\mathcal N_{\mathcal F}$ the curvature tensor of normal bundle connection of a single fiber $\mathcal F_u$. Then*

$$\begin{aligned}g^{\mathcal F}(\mathcal N_{\mathcal F}(\eta^v,\zeta^v)X^h,Y^h)=&\,g^{\mathcal B}(\hat R^{\mathcal F}(\eta,\zeta)X,Y)+\\ &\frac14 g^{\mathcal B}(\hat R^{\mathcal F}(\xi,\eta)X,\hat R^{\mathcal F}(\xi,\zeta)Y)-\frac14 g^{\mathcal B}(\hat R^{\mathcal F}(\xi,\eta)Y,\hat R^{\mathcal F}(\xi,\zeta)X).\end{aligned}$$

The latter corollary means that the extrinsic geometry of the fibers is defined by the curvature of the fiber bundle connection.

2. Unit sphere bundle and unit sections.

Denote by $\mathcal E_1\subset\mathcal E$ a subbundle defined by the equation $g^{\mathcal F}(\xi,\xi)=1$. The fibers of $\mathcal E_1$ are unit spheres and $\pi:\mathcal E_1\to\mathcal B$ is called *a unit sphere bundle* over $\mathcal B$. The $\mathcal E_1$ is a hypersurface in $\mathcal E$ with the Sasaki-type pull-back metric. At each point $(u,\xi)\in\mathcal E_1$, the ξ^v is a unit normal for $\mathcal E_1\subset\mathcal E$. Consider a unit section $\xi:\mathcal B\to\mathcal E_1$ as a (local) imbedding of the base into $(\mathcal E_1,g^{\mathcal E})$.

Definition 2 *Let $(\mathcal E_1,g^{\mathcal E})$ be a unit vector bundle with the Sasaki type metric. A unit section $\xi:\mathcal B\to(\mathcal E_1,g^{\mathcal E})$ is called minimal (totally geodesic) if $\xi(\mathcal B)$ is a minimal (totally geodesic) submanifold.*

For a given unit section ξ, the differential $\xi_* : T\mathcal{B} \to T\mathcal{E}_1$ acts by

$$\xi_* X = X^h + (\nabla^{\mathcal{F}}_X \xi)^v.$$

Define a point-wise linear operator $A_\xi : T_u\mathcal{B} \to \mathcal{F}_u$ and its conjugate $A^t_\xi : \mathcal{F}_u \to T_u\mathcal{B}$ by

$$A_\xi X = -\nabla^{\mathcal{F}}_X \xi, \qquad g^{\mathcal{B}}(A^t_\xi \eta, X) = g^{\mathcal{F}}(A_\xi X, \eta).$$

Then the tangent and normal vector fields on $\xi(\mathcal{B})$ can be described as follows:

$$\begin{aligned} &\tilde X \in T\xi(\mathcal{B}) && \text{iff} \quad \tilde X = X^h - (A_\xi X)^v; \\ &\tilde N \in T^\perp \xi(\mathcal{B}) && \text{iff} \quad \tilde N = (A^t_\xi \eta)^h + \eta^v \quad (\eta \perp \xi). \end{aligned}$$

By using the standard computations one can prove the following Lemma which is similar to the one for the unit sections of the unit tangent bundle [8].

Lemma 4 *Let ξ be a smooth unit section of a smooth unit vector bundle $\mathcal{E}_1$ with the Sasaki-type metric $g^{\mathcal{E}}$. Denote by $\tilde\Omega_{\tilde N}$ the second fundamental form of $\xi(\mathcal{B}) \subset \mathcal{E}_1(\mathcal{B})$ with respect to the normal vector field $\tilde N = (A^t_\xi \eta)^h + \eta^v$ $(\eta \perp \xi)$. Then for any $X, Y \in \mathfrak{X}(\mathcal{B})$,*

$$\begin{aligned} &\tilde\Omega_{\tilde N}(\xi_* X, \xi_* Y) = \\ &\quad -\frac{1}{2} g^{\mathcal{F}}\Big((\nabla^{\mathcal{F}}_X A_\xi)Y + (\nabla^{\mathcal{F}}_Y A_\xi)X + A_\xi\big(\hat R^{\mathcal{F}}(\xi, A_\xi X)Y + \hat R^{\mathcal{F}}(\xi, A_\xi Y)X\big), \eta\Big) \end{aligned}$$

where $(\nabla^{\mathcal{F}}_X A_\xi)Y = \nabla^{\mathcal{F}}_X (A_\xi Y) - A_\xi(\nabla^{\mathcal{B}}_X Y)$.

As a consequence, we can easily prove the following statement.

Lemma 5 *The submanifold $\xi(\mathcal{B}) \subset (\mathcal{E}_1, g^{\mathcal{E}})$ is totally geodesic if and only if the section ξ satisfies*

$$\begin{aligned} &(\nabla^{\mathcal{F}}_X A_\xi)Y + (\nabla^{\mathcal{F}}_Y A_\xi)X + \\ &\qquad A_\xi\big(\hat R^{\mathcal{F}}(\xi, A_\xi X)Y + \hat R^{\mathcal{F}}(\xi, A_\xi Y)X\big) - 2g^{\mathcal{F}}(A_\xi X, A_\xi Y)\xi = 0 \end{aligned} \tag{3}$$

for any $X, Y \in \mathfrak{X}(\mathcal{B})$.

The equation (3) represents over-definite system of PDEs with respect to the section ξ and involves the bundle connection of $\mathcal{E}$ and the Riemannian connection of $T\mathcal{B}$. The first question is, if the equation could have a solution? In the next section we give positive answer to this question.

2.1 A unit circle bundle over a surface.

The simplest non-trivial case for the equation (3) is the case $n = \dim \mathcal{B} = 2$ and $p = \dim \mathcal{F} = 2$. For this case we will use the terms "*a 2-vector bundle over a surface*" and "*a unit circle bundle over a surface*". In this section we will conduct all our calculations over local trivializing chart of the vector bundle $\pi : \mathcal{E} \to \mathcal{B}$ without special emphasis.

We begin with the case of *flat bundle connection* in $\mathcal{E}$. The base Riemannian 2-manifold, nevertheless, could have zero or non-zero Gaussian curvature, i.e. could be locally isometric/non-isometric to the Euclidean plane.

Theorem 1 *Let $\pi : \mathcal{E} \to \mathcal{B}$ be 2-vector bundle with flat bundle connection over a surface $\mathcal{B}$. Let ξ be a unit totally geodesic local section of a unit circle bundle $\pi : \mathcal{E}_1 \to \mathcal{B}$ with the Sasaki-type metric.*

- *If $\mathcal{B}$ is not locally Euclidean, then ξ is arbitrary parallel unit section;*
- *If $\mathcal{B}$ is locally Euclidean with the Cartesian coordinates (x, y), then ξ makes the angle $\theta(x, y) = ax + by + c$ with arbitrary parallel unit section.*

Proof. Since the bundle connection of $\mathcal{E}$ is flat, there is a pair of orthonormal sections which are parallel with respect to the bundle connection. Denote by s_1 and s_2 these sections. Then any unit section can be given by

$$\xi = \cos\theta\, s_1 + \sin\theta\, s_1.$$

Denote $\xi^\perp = -\sin\theta\, s_1 + \cos\theta\, s_2$. Then $A_\xi X = -\nabla^{\mathcal{F}}_X \xi = -X(\theta)\xi^\perp, \quad \nabla^{\mathcal{F}}_X \xi^\perp = -X(\theta)\xi$ and we have

$$(\nabla^{\mathcal{F}}_X A_\xi)Y = \nabla^{\mathcal{F}}_X (A_\xi Y) - A_\xi(\nabla^{\mathcal{B}}_X Y) = -\nabla^{\mathcal{F}}_X (Y(\theta)\xi^\perp) + (\nabla^{\mathcal{B}}_X Y)(\theta)\xi^\perp = \\ - X(Y(\theta)\xi^\perp + X(\theta)Y(\theta)\xi + (\nabla^{\mathcal{B}}_X Y)(\theta)\xi^\perp.$$

By definition, $X(Y(\theta)) - (\nabla^{\mathcal{B}}_X Y)(\theta) = Hess^{\mathcal{B}}_\theta(X, Y)$ and hence

$$(\nabla^{\mathcal{F}}_X A_\xi)Y = -Hess^{\mathcal{B}}_\theta(X, Y)\xi^\perp + X(\theta)Y(\theta)\xi = (\nabla^{\mathcal{F}}_Y A_\xi)X.$$

Then, the equation (3) takes the form $-2\, Hess^{\mathcal{B}}_\theta(X, Y)\xi^\perp = 0$. As a conclusion, the section ξ is totaly geodesic iff

$$Hess^{\mathcal{B}}_\theta(X, Y) = X^i Y^k \left(\frac{\partial^2\theta}{\partial u^i u^k} - \overset{\mathcal{B}}{\Gamma}{}^m_{ik} \frac{\partial\theta}{\partial u^m}\right) = 0, \tag{4}$$

where $\overset{\mathcal{B}}{\Gamma}{}^m_{ik}$ mean the Christoffel symbols of the base manifold Riemannian metric.

- If $\theta = const$, then ξ is a parallel local unit section and $\xi(\mathcal{B})$ is totally geodesic independently on geometry of the base manifold;
- If $d\theta \neq 0$, then $\xi(\mathcal{B})$ is totally geodesic if $d\theta$ is a nonzero parallel 1-form on the base manifold. In other words, $\mathcal{B}$ admits a parallel local vector field (namely, the $\operatorname{grad}\theta$). In this case the Gaussian curvature of the base manifold is zero, i.e. $\mathcal{B}$ is locally Euclidean. Choose the Cartesian coordinates

(x, y) on the base manifold. Then the line element of $\mathcal{E}_1$ takes the form (see (2))

$$d\tilde{s}^2 = dx^2 + dy^2 + d\theta^2$$

and hence $(\mathcal{E}_1, g^{\mathcal{E}})$ is locally Euclidean (while topologically $\mathcal{E}_1 = E^2 \times S^1$). So we have

$$\theta = ax + by + c$$

as a solution to the equation (4). Geometrically, we get a linear angle function between the totally geodesic local unit section ξ and a parallel unit section.

■

Consider the case of *non-flat bundle connection.* Introduce the *bi-sectional curvature* function of $\mathcal{E}$ by

$$\varkappa = \frac{g^{\mathcal{F}}\big(R^{\mathcal{F}}(X,Y)\xi, \eta\big)}{|X \wedge Y|_{g^{\mathcal{B}}} \cdot |\xi \wedge \eta|_{g^{\mathcal{F}}}},$$

where $X, Y \in \mathfrak{X}(\mathcal{B})$ and $\xi, \eta \in \mathfrak{S}(\mathcal{B})$. Up to a sign, it is nothing else but the Gaussian curvature in the case of $\mathcal{E} = TM^2$ and the Gaussian torsion in the case of normal bundle $\mathcal{E} = T^{\perp}F^2$ of a submanifold in the Riemannian M^4. If both frames are orthonormal, then $\varkappa = g^{\mathcal{F}}\big(R^{\mathcal{F}}(X,Y)\xi, \eta\big)$.

If $n \geq p$, then the kernel of A_ξ is non-empty by the dimension reasons. Denote

$$\mathcal{Z} = \ker A_\xi \subset \mathfrak{X}(\mathcal{B}), \quad \mathfrak{I} = \operatorname{im} A_\xi \subset \xi^{\perp} \subset \mathfrak{S}(\mathcal{B})$$

If $\mathcal{Z}_q = T_q\mathcal{B}$ for all $q \in \mathcal{B}$, then the bundle connection is flat. In general, $T_q\mathcal{B} = \mathcal{Z}_q \oplus \mathcal{Z}_q^{\perp}$. In general setting, for each given section ξ we have two complementary distributions $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ on $\mathcal{B}$. The following statement simplifies Lemma 4.

Lemma 6 *Let ξ be a unit local section of a unit circle bundle $\pi : \mathcal{E}_1 \to \mathcal{B}$ over a surface. Suppose the bundle connection of $\pi : \mathcal{E} \to \mathcal{B}$ is non-flat. Denote by (e_1, e_2) the orthonormal local tangent frame on $\mathcal{B}$ such that $\mathcal{Z} = Span\{e_1\}$, $\mathcal{Z}^{\perp} = Span\{e_2\}$. Then the second fundamental form $\tilde{\Omega}$ of $\xi(\mathcal{B}) \subset \mathcal{E}_1$ satisfies*

$$[\tilde{\Omega}(e_i, e_j)] = \begin{bmatrix} -k_1 \sin(\alpha/2) & \frac{1}{2}(e_1(\alpha) - \varkappa) \\ \frac{1}{2}(e_1(\alpha) - \varkappa) & \frac{1}{2}e_2(\alpha)\cos(\alpha/2) \end{bmatrix} \tag{5}$$

where $\varkappa$ is a bi-sectional curvature of $\mathcal{E}$, $\alpha/2$ is the angle between the unit normal $\tilde{n}$ of a hypersurface $\xi(\mathcal{B}) \subset \mathcal{E}_1$ and the vertical (one-dimensional) subspace and k_1 is a geodesic curvature of the field e_1.

Proof. Take a unit section $\eta = \xi^{\perp}$. Take (e_1, e_2) as in a hypothesis. Then we may put

$$A_\xi e_1 = 0, \quad A_\xi e_2 = -\lambda\eta.$$

Then

$$\tilde{e}_1 = e_1^h, \qquad \tilde{e}_2 = \frac{1}{\sqrt{1+\lambda^2}}(e_2^h + \lambda\eta^v) \tag{6}$$

form the orthonormal tangent frame on $\xi(\mathcal{B})$. A unit normal vector field $\tilde{n}$ on $\xi(\mathcal{B})$ is

$$\tilde{n} = \frac{1}{\sqrt{1+\lambda^2}}(-\lambda e_2^h + \eta^v). \tag{7}$$

Denote by $\alpha/2$ the angle between $\tilde{n}$ and the vertical distribution. Then

$$\cos(\alpha/2) = \frac{1}{\sqrt{1+\lambda^2}}, \quad \sin(\alpha/2) = \frac{\lambda}{\sqrt{1+\lambda^2}}, \quad \lambda = \tan(\alpha/2)$$

and hence

$$\tilde{e}_1 = e_1^h, \quad \tilde{e}_2 = \cos(\alpha/2)e_2^h + \sin(\alpha/2)\,\eta^v, \quad \tilde{n} = -\sin(\alpha/2)\,e_2^h + \cos(\alpha/2)\,\eta^v.$$

Define k_1 and k_2 by $\nabla^{\mathcal{B}}_{e_1} e_1 = k_1 e_2, \quad \nabla^{\mathcal{B}}_{e_2} e_2 = k_2 e_1$. Then by Lemma 2,

$$\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_1 = (\nabla^{\mathcal{B}}_{e_1} e_1)^h = k_1 e_2^h, \quad \tilde{\nabla}_{\tilde{e}_1}\tilde{e}_2 = \tfrac{1}{2}k_1\cos(\alpha/2)e_1^h + \tfrac{1}{2}(e_1(\alpha) - \varkappa)\tilde{n},$$

$$\tilde{\nabla}_{\tilde{e}_2}\tilde{e}_2 = (k_1\cos(\alpha/2) - \sin\alpha\varkappa)e_1^h + \tfrac{1}{2}e_2(\alpha)\cos(\alpha/2)\tilde{n} + \sin^2(\alpha/2)\xi^v.$$

Hence

$$\tilde{\Omega}(\tilde{e}_1, \tilde{e}_1) = -\sin(\alpha/2)k_1, \quad \tilde{\Omega}(\tilde{e}_1, \tilde{e}_2) = \tfrac{1}{2}(e_1(\alpha) - \varkappa),$$
$$\tilde{\Omega}(\tilde{e}_2, \tilde{e}_2) = \cos(\alpha/2)\,e_2(\alpha/2),$$

which completes the proof.

■

Corollary 2 *Let ξ be a unit section of a unit circle bundle $\pi : \mathcal{E}_1 \to \mathcal{B}$ over a surface. Suppose the bundle connection of $\pi : \mathcal{E} \to \mathcal{B}$ is non-flat. Denote by H the mean curvature of $\xi(\mathcal{B})$ and by $\tilde{n}_h$ the horizontal projection of the unit normal vector field on $\xi(\mathcal{B}) \subset \mathcal{E}_1$. Then*

$$\tilde{H} = -\frac{1}{2}\widetilde{div}(\tilde{n}) = -\frac{1}{2}div(\tilde{n}_h).$$

Proof. Indeed, the equality $\tilde{H} = -\frac{1}{2}\widetilde{div}(\tilde{n})$ follows from the definitions. As for the second equality, we have $\tilde{n}_h = -\sin(\alpha/2)e_2$ and

$$\nabla^{\mathcal{B}}_{e_1}\tilde{n}_h = -\tfrac{1}{2}\cos(\alpha/2)e_1(\alpha)e_2 + k_1\sin(\alpha/2)e_1,$$
$$\nabla^{\mathcal{B}}_{e_2}\tilde{n}_h = -\tfrac{1}{2}\cos(\alpha/2)e_2(\alpha)e_2 - k_2\sin(\alpha/2)e_1.$$

Therefore,

$$\tilde{\Omega}(\tilde{e}_1, \tilde{e}_1) = -\langle \nabla^{\mathcal{B}}_{e_1}\tilde{n}_h, e_1\rangle, \quad \tilde{\Omega}(\tilde{e}_2, \tilde{e}_2) = -\langle \nabla^{\mathcal{B}}_{e_2}\tilde{n}_h, e_2\rangle \text{ and } \tilde{H} = -\tfrac{1}{2}div(\tilde{n}_h).$$

■

Now we can prove the main result of the section.

Theorem 2 *Let $\pi : \mathcal{E}_1 \to \mathcal{B}$ be a unit circle bundle with the Sasaki-type metric over a surface. Suppose the bundle connection of $\pi : \mathcal{E} \to \mathcal{B}$ is non-flat. Then $\mathcal{E}_1$ admits a local totally geodesic unit section if and only if $\mathcal{B}$ is locally isometric to*

$$(M^2, ds^2 = du^2 + \sin^2\alpha(u)dv^2)$$

and the bi-sectional curvature of $\mathcal{E}$ satisfies $\varkappa = \dot\alpha(u)$.

Proof. Suppose $\mathcal{E}_1$ admits a local totally geodesic section. Since $\varkappa \neq 0$, the section is non parallel and hence the surface $\xi(\mathcal{B})$ is not horizontal. Therefore, there is a trivializing neighborhood $\mathcal{U} \subset \mathcal{B}$ such that $\alpha|_{\mathcal{U}} \neq 0$. Restrict our considerations to $\mathcal{U}$ and choose (e_1, e_2) as in Lemma 6. Then $\xi(\mathcal{U})$ is totally geodesic if, particularly,

$$e_2(\alpha) \equiv 0, \quad k_1 \equiv 0.$$

Thus, the trajectories of e_1 are geodesics and the angle function α is constant along e_2.

Choose the semi-geodesic coordinate system (u, v) on $\mathcal{U}$ such that

$$\partial_u = e_1, \quad \partial_v = f(u, v)\, e_2,$$

where $f(u, v)$ is some non-zero function. Then

$$ds^2 = du^2 + f^2 dv^2.$$

With respect to these coordinates, the conditions on α take the form

$$\partial_v\alpha = 0, \quad \partial_u\alpha = \varkappa.$$

Hence

$$\varkappa = \dot\alpha.$$

On the other hand,

$$\mathcal{R}^{\mathcal{F}}(e_1, e_2)\xi = (\nabla^{\mathcal{F}}_{e_2} A_\xi)e_1 - (\nabla^{\mathcal{F}}_{e_1} A_\xi)e_2 = \\ - A_\xi(\nabla^{\mathcal{B}}_{e_2} e_1) - \nabla^{\mathcal{F}}_{e_1}(A_\xi e_2) + A_\xi(\nabla^{\mathcal{B}}_{e_1} e_2) = (e_1(\lambda) - k_2\lambda)\eta.$$

Since $\lambda = \tan(\alpha/2)$, we have

$$\dot\alpha = \frac{\dot\alpha}{2\cos^2(\alpha/2)} - \tan(\alpha/2)k_2.$$

Since $\alpha = \alpha(u)$, we see that $k_2 = k_2(u)$. With respect to the chosen coordinate system

$$k_2 = -\frac{\partial_u f}{f}.$$

Hence, $f(u, v) = a(v)h(u)$ and after the parameter change

$$ds^2 = du^2 + h^2(u)\, dv^2.$$

Thus we have the equation on $h(u)$

$$\dot\alpha = \frac{\dot\alpha}{2\cos^2(\alpha/2)} - \tan(\alpha/2)(-\frac{\dot h}{h}),$$

with a general solution

$$h(u) = C\sin\alpha.$$

After the parameter v rescaling, we come to

$$ds^2 = du^2 + \sin^2\alpha\, dv^2.$$

Conversely, let $\pi : \mathcal{E}_1 \to \mathcal{B}$ is a unit sphere bundle over the Riemanian manifold $\big(\mathcal{B}, ds^2 = du^2 + \sin^2\alpha(u)\, dv^2\big)$ with the bi-sectional curvature of the bundle connection $\varkappa = \dot\alpha(u)$. Let us show, that there is a local section ξ which satisfies

$$\nabla^{\mathcal{F}}_{\partial_u}\xi = 0, \quad \nabla^{\mathcal{F}}_{\partial_v}\xi = 2\sin^2(\alpha/2)\eta$$

Take an arbitrary orthonormal sections $\{s_1, s_2\}$. Then

$$\xi = \ \cos\theta\, s_1 + \sin\theta\, s_2, \quad \eta = -\sin\theta\, s_1 + \cos\theta\, s_2,$$

where θ is some smooth function. Then

$$\big\langle\nabla^{\mathcal{F}}_{\partial_u}\xi, \eta\big\rangle = \partial_u\theta + \gamma^2_{1|1} = 0, \quad \big\langle\nabla^{\mathcal{F}}_{\partial_v}\xi, \eta\big\rangle = \partial_v\theta + \gamma^2_{1|2} = 2\sin^2(\alpha/2)$$

or

$$\partial_1\theta = \gamma^1_{2|1}, \quad \partial_2\theta = \gamma^1_{2|2} + 2\sin^2(\alpha/2).$$

Then the integrability condition takes the form

$$\partial_2\gamma^1_{2|1} - \partial_1\gamma^1_{2|2} = 2\sin(\alpha/2)\cos(\alpha/2)\dot\alpha = \sin(\alpha)\dot\alpha.$$

In the left hand side we have $g^{\mathcal{F}}\big(R^{\mathcal{F}}(\partial_1,\partial_2)s_1, s_2\big)$. By definition,

$$\frac{g^{\mathcal{F}}\big(R^{\mathcal{F}}(\partial_1,\partial_2)s_1, s_2\big)}{\sin\alpha} = \varkappa.$$

Hence, the condition $\dot\alpha = \varkappa$ provides the integrability.

■

Direct computation implies the following assertion.

Corollary 3 *Suppose $\pi : \mathcal{E}_1 \to \mathcal{B}$ is a unit circle bundle over a surface. Suppose the bundle connection of $\pi : \mathcal{E} \to \mathcal{B}$ is non-flat and $\mathcal{E}_1$ admits a totally geodesic local section. Then the base manifold Gaussian curvature K and the bi-sectional bundle curvature $\varkappa$ satisfy*

$$K = \varkappa^2 - \cot\alpha(u)\dot\varkappa. \tag{8}$$

Examples.

3.1 The unit tangent bundle.

In this case one can find the totally geodesic section explicitly.

Theorem 3 *[11] Let (M^2, g) be a Riemannian manifold with a sign -preserving non-zero Gaussian curvature K. Then M admits a local totally geodesic unit vector field ξ if and only if there is a local parametrization of M with respect to which the line element takes the form*

$$ds^2 = du^2 + \sin^2 \alpha(u)\, dv^2,$$

where $\alpha(u)$ is a solution of the differential equation

$$\frac{d\alpha}{du} = 1 - \frac{a+1}{\cos\alpha}.$$

The corresponding local unit vector field ξ is of the form

$$\xi = \cos(av + \omega_0)\, \partial_u + \frac{\sin(av + \omega_0)}{\sin \alpha(u)}\, \partial_v,$$

where $a, \omega_0 = const.$

It is worthwhile to mention that Gaussian curvature K of the metric from the Theorem 3 is

$$K = \frac{d\alpha}{du} = 1 - \frac{a+1}{\cos\alpha}$$

and after evident reparametrization, the metric takes the form

$$ds^2 = \frac{1}{K(\alpha)}\, d\alpha^2 + \sin^2\, \alpha\, dv^2.$$

The curvature K could be non-zero constant iff $a = -1$ and $K = 1$. In this case, the integral trajectories of the totally geodesic unit vector field ξ are stereographically equivalent to the pencil of parallel straight lines on the plane [11].

In general, the integral trajectories of the totally geodesic unit vector fields on M^2 are conformally equivalent to the trajectories of the totally geodesic unit vector field on the plane E^2.

3.2 The unit normal bundle

In this case the bi-sectional bundle curvature is the same as the Gaussian torsion $\varkappa_\Gamma$. In [12] it was proved, that *every analytic 2-dimensional metric admits a local isometric immersion into E^4 as analytic surface with prescribed analytic Gaussian torsion.* So, if we take arbitrary monotonic smooth function $0 < \alpha(u) < \pi$ and put $\varkappa_\Gamma = \dot\alpha(u)$, then there is a surface in E^4 with unit normal bundle satisfying the conditions of Theorem 2. For example, if we take $\alpha = c\,u$ and $\varkappa_\Gamma = \dot\alpha = c$, then the Gaussian curvature of the base $K = c^2$ and we get a local constant curvature surface with constant Gaussian torsion in E^4 with some local totally geodesic unit section. Observe, that in this case $T_1^{\perp}F^2$ is a space of constant curvature $\frac{c^2}{4}$.

Another surface which satisfies the conditions of Theorem 2 is the Veronese surface $V^2(r) \subset S^4(R) \subset E^5$ given by

$$y = \left\{ \frac{1}{\sqrt{3}}x_1x_3, \frac{1}{\sqrt{3}}x_2x_3, \frac{1}{\sqrt{3}}x_1x_2, \frac{1}{2\sqrt{3}}(x_1^2 - x_2^2), \frac{1}{6}(x_1^2 + x_2^2 - 2x_3^2) \right\},$$

where $x_1^2 + x_2^2 + x_3^2 = r^2$. This is a surface in a sphere of radius $R = \frac{r^2}{3}$. The V^2 has constant Gaussian curvature K and constant Gaussian torsion $\varkappa_\Gamma$, namely

$$K = \frac{3}{r^4}, \quad \varkappa_\Gamma = \frac{6}{r^4} = 2K.$$

The necessary condition (8) is fulfilled if $K = \frac{1}{4}$, that is when $r = \sqrt[4]{12}$. In this case

$$K = \frac{1}{4}, \quad \varkappa_\Gamma = \frac{1}{2}.$$

By passing to spherical coordinates

$$x_1 = r\sin(u/2)\cos(v/2), \quad x_2 = r\sin(u/2)\sin(v/2), \quad x_3 = r\cos(u/2),$$

the first fundamental form of the Veronese surface takes the form (for $r = \sqrt[4]{12}$)

$$ds^2 = du^2 + \sin^2(u/2)\,dv^2.$$

Evidently, $\alpha(u) = \frac{u}{2}$ and exactly $\dot\alpha_u = \varkappa_\Gamma = \frac{1}{2}$. Remark that $T_1^{\perp}(V^2(\sqrt[4]{12})$ is of constant sectional curvature $\frac{1}{16}$.

3.3 Minimal unit normal sections

The example below show that there is a minimal but not totaly geodesic global section of a unit normal bundle of a surface in the Euclidean space.

Propozition 1 *The graph of complex curve $w = e^z$ admits a global unit normal vector field which provides a global minimal but not totally geodesic embedding of the 2-plane into the unit normal bundle of the curve.*

Proof. The complex curve $w = e^z$ is a surface in E^4 given by

$$\vec r = \{x, y, e^x\cos y, e^x\sin y\},$$

The tangent (non-orthonormal) and normal frames are given by

$$\partial_x = \{1, 0, e^x\cos y, e^x\sin y\}, \quad \partial_y = \{0, 1, -e^x\sin y, e^x\cos y\},$$

$$n_1 = \frac{1}{\sqrt{1+e^{2x}}}\{-e^x\cos y, e^x\sin y, 1, 0\},$$

$$n_2 = \frac{1}{\sqrt{1+e^{2x}}}\{-e^x \sin y, -e^x \cos y, 0, 1\}.$$

Put $\xi = n_1$ and $\eta = n_2$. Then the frame

$$e_1 = -\frac{1}{\sqrt{1+e^{2x}}}\,\partial_x, \quad e_2 = \frac{1}{\sqrt{1+e^{2x}}}\,\partial_y,$$

meets the requirements of the Lemma 6 with

$$\tan(\alpha/2) = \lambda = \frac{e^{2x}}{(1+e^{2x})^{3/2}}.$$

We have

$$\tilde n_h = -\frac{\lambda}{\sqrt{1+\lambda^2}}\,e_2$$

and then

$$\big\langle \nabla_{e_1}\tilde n_h, e_1\big\rangle = 0, \quad \big\langle \nabla_{e_2}\tilde n_h, e_2\big\rangle = 0,$$

that is $\tilde H = 0$.

The Gaussian curvature and Gaussian torsion are of the form

$$K = -\varkappa_\Gamma = -2\frac{e^{2x}}{(1+e^{2x})^3}.$$

Since $\alpha = 2\arctan\lambda$, then

$$e_1(\alpha) = 2\frac{e_1(\lambda)}{1+\lambda^2} = -2\frac{e^{2x}(e^{2x}-2)}{1+3e^{2x}+4e^{4x}+e^{6x}} \neq \varkappa_\Gamma.$$

■

The next example is interesting itself. Consider the tangent bundle TM^2 with the Sasaki metric. It is known that the fibers are totally geodesic and intrinsically flat submanifolds in TM^2. The Corollary 1 implies that the Gaussian torsion $\varkappa_\Gamma$ of a single fiber T_xM^2 is equal to the Gaussian curvature of the base at x and hence is constant.

Propozition 2 *Denote by T_xM^2 a fiber of TM^2 with Sasaki metric. Let $\xi = X_x^h$ be a unit normal vector field on T_xM^2, where $X_x \in T_xM^2$. Then ξ maps the fiber into minimal submanifold in the unit normal bundle of the fiber.*

Proof. Take a fiber T_xM^2. Then at each point $\xi \in T_xM^2$, the tangent frame of T_xM^2 consists of η_x^v, ζ_x^v and the normal frame consists of X_x^h, Y_x^h, where $(\eta_x, \zeta_x, X_x, Y_x) \in T_xM^2$. As the fibers are totally geodesic, the curvature tensor of the normal bundle connection of the fiber is defined by the curvature tensor component of TM^2 of the form

$$\tilde g_{(x,\xi)}\big(\tilde R(\eta^v, \zeta^v)X^h, Y^h\big) = g_x\big(R(\eta,\zeta)X, Y\big).$$

Hence, the Gaussian torsion of the fiber is

$$\varkappa_\Gamma(T_xM^2) = K(x),$$

where $K(x)$ is the Gaussian curvature of M^2 at $x \in M^2$.

Let (η, ζ) and (X, Y) be orthonormal frames in T_xM^2 oriented in such a way that

$$g_x\big(R(\eta,\zeta)X, Y\big) = K(x).$$

By applying Lemma 2 to the case of tangent bundle, we get

$$\tilde\nabla_{\eta^v} X^h = \frac{1}{2}\Big(R(\xi,\eta)X\Big)^h.$$

Decompose

$$\xi = \xi^1\eta + \xi^2\zeta.$$

As the fiber metric is Euclidean,

$$\tilde n_h = -\frac{K(x)\{-\xi^2, \xi^1\}}{\sqrt{1 + K^2(x)((\xi^1)^2 + (\xi^2)^2)}}.$$

and hence

$$-div\,(\tilde n_h) = \partial_{\xi^1} Z^1 + \partial_{\xi^2} Z^2 = \frac{K^3(x)}{(1 + K^2(x)|\xi|^2)^{3/2}}(-\xi^2\xi^1 + \xi^1\xi^2) = 0.$$

So we see, that any unit normal vector field on the fiber T_xM^2 of the form $\xi = X^h_x$ is minimal in the unit normal bundle of the fiber.

A single fiber $T_xM^2 \subset TM^2$ does not admit a totally geodesic unit normal section of its normal bundle if the base manifold have non-zero curvature at the corresponding point. Indeed, the single has zero Gaussian curvature while the Gaussian torsion of the fiber is equal to the Gaussian curvature $K(x) \neq 0$ of the base manifold at the corresponding point (and hence is constant along the fiber) and we come to a contradiction with (8).

References

1. Borisenko A., Yampolsky A. Riemannian geometry of bundles.// Uspehi Mat. Nauk. 1991. - **26/6**. - P. 51-95; Engl. transl.: Russian Math. Surveys, 1991. - **46/6**. - P. 55-106.

2. Gluck H., Ziller W. On the volume of a unit vector field on the threesphere.// Comm. Math. Helv., 1986. - **61**. - P. 177-192.

3. Gil-Medrano O.,Llinares-Fuster E. Minimal unit vector fields. // Tôhonku Math. J., 2002. - **54**. - P. 71-84.

4. Yampolsky A. On the mean curvature of a unit vector field. // Math. Publ. Debrecen, 2002. - **60/1-2**. - P. 131-155.

5. Yampolsky A. A totally geodesic property of Hopf vector fields. // Acta Math. Hungar., 2003. - **101/1-2**. - P. 93-112.

6. C.M. Wood. Harminic sections and equivariant harmoc maps. // Manuscripta Math., 1997, - **94**. - P. 1-13.

7. C.M. Wood. Harmonic sectios of homogeneous fiber bundles // Diff. Geom. Appl., 2003. - **19**. - P. 193-210

8. Yampolsky A. On special tipes of minimal and totally geodesic unit vector fields. // Seventh Int. Conf. on Geometry, Integrabilityand Quantization, June 2 – 10, 2005, Varna, Bulgaria, P. 290-304

9. Dombrowski P. *On the geometry of tangent bundle*, J. Reine Angew. Math., 1962, **210**/1-2 , 73–88.

10. Kowalski O. *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math. 1971, **250** , 124–129.

11. Yampolsky A., Full Description of Totally Geodesic Unit Vector Fields on 2-dimensional Riemannian Manifold. // Matematicheskaya Fizika, Analiz, Geometriya 2004. - **11/3**. - P. 355-365.

12. Аминов Ю.А. О поверхностях в E^4 со знакопостоянным гауссовым кручением.// Укр. геом. сб., 1988, - вып. **31**. - P. 3-14.

13. Yampolsky A.A. On intrinsinc geometry of a unit vector field. // Comment. Math. Univ. Carol., 2002, - **43/2**. - P. 131-155.

14. Boeckx E., Vanhecke L. Harmonic and minimal radial vector fields. // Acta Math. Huhgar., 2001, - **90**. - P. 317-331.

15. Boeckx E., Vanhecke L. Harmonic and minimal vector fields on tangent and unit tangent bundles. // Differential Geom. Appl., 2000, - **13**. - P. 77-93.

16. Gil-Medrano O.,Llinares-Fuster E. Second variation of volume and energy of vector fields. Stability of Hopf vector field. // Math. Ann., 2001, - **320**. - P. 531-545.

17. Gil-Medrano O. Relationship between volume and energy of vector fields. // Diff. Geom. Appl., 2001. - **15**. - P. 137-152.

18. González-Dávila J.C., Vanhecke L. Examples of minimal unit vector fields.// Ann Global Anal. Geom., 2000. - **18**. - P. 385-404.

19. Kobayashi S., Nomizu K. Foundations of differential geometry, Vol. 1, Interscience Publ., 1963.

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